TL-subalgebras and TL-ideals of BCK-algebras

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Abstract—In this paper, the concepts of TL-subalgebras, TL-ideals and TL-implicative ideals of BCK-algebra are introduced. A necessary and sufficient condition for a L-subset of BCK-algebra to be a L-subalgebra (ideal, implicative ideal) is stated, and images and inverse-images of TL-subalgebra under BCK-algebra homomorphism are studied. Also, several characterizations of TL-ideals (implicative ideals) are given. Where T is an arbitrary infinitely \( \vee \)-distributive t-norm on a given complete Brouwerian lattice L.

I. INTRODUCTION

A BCK-algebra is an important class of logical algebras introduced by Iséki [13] and was extensively investigated by several researchers. Zadeh [14] introduced the notion of fuzzy sets. It was first applied to the theory of groups by Rosenfeld [1]. Since then, many authors introduced fuzzy subring and fuzzy ideals [2-4], fuzzy subalgebras [5,6], and so on. Especially, the concepts of TL-subrings, TL-ideals [7,8] and T-congruence L-relations [9] were proposed, and their properties were carefully studied to a certain extent. Xi [10] applied the concept of fuzzy set to BCK-algebras. After that Jun and Meng investigated further properties of fuzzy BCK-algebras and fuzzy ideals [11,12].

In this paper, using a general infinitely \( \vee \)-distributive t-norm T on a complete Brouwerian lattice L, we shall introduce the concepts of TL-subalgebras and TL-ideals of BCK-algebras and obtain some results. Throughout this paper, unless otherwise stated, L always represents any given complete Brouwerian lattice with maximal element 1 and minimal element 0; T any given infinitely \( \vee \)-distributive t-norm on L.

II. PRELIMINARIES

Definition 2.1[7]. A binary operation T on L is called a t-norm on L if it satisfies the following conditions:
(1) \( (aTb)Tc = aT(bTc) \) for all \( a,b,c \in L \);
(2) \( aTb = bTa \) for all \( a,b \in L \);
(3) \( b \leq c \Rightarrow aTb \leq aTc \) for all \( a,b,c \in L \);
(4) \( aT1 = a \) for all \( a \in L \).

A t-norm T on L is said to be \( \vee \)-distributive if
\[
 aT(b \vee c) = (aTb) \vee (aTc)
\]
for all \( a,b,c \in L \).

Infinitely \( \vee \)-distributive if
\[
 aT(\bigvee_{i \in I} b_i) = \bigvee_{i \in I} (aTb_i)
\]
for all \( a,b \in L \), \( i \in I \), where I is any nonempty index set.

By an L-subset of L, we mean a mapping from G into L. The set of all L-subsets of L is called the L-power set of L and denoted by \( L^G \).

Definition 2.2[13]. An algebraic system \((G; \ast, 0)\) of type \((2,0)\) is said to be a BCK-algebra if it satisfies: for all \( x, y, z \in G \),
\[
 (B-1) \ (x \ast y) \ast (x \ast z) = (x \ast y) \ast z = x \ast (y \ast z),
\]
\[
 (B-2) \ (x \ast (x \ast y)) = y = 0,
\]
\[
 (B-3) \ x \ast x = 0,
\]
\[
 (B-4) \ x \ast y = y \ast x = 0 \Rightarrow x = y.
\]

In a BCK-algebra G, we can define a partial ordering \( \leq \) by putting \( x \leq y \) if and only if \( x \ast y = 0 \).

Definition 2.3[10]. A subset I of a BCK-algebra G is called an ideal of G if
\[
 (I_1) \ 0 \in I;
\]
\[
 (I_2) \ \text{for any } x, y \in G, \ x \ast y \in I, \ y \in I \Rightarrow x \in I.
\]

A non-empty subset I of G is called an implicative ideal if it satisfies \((I_1)\) and \((I_2)\): \( x \ast I \subseteq I \) whenever \( (x \ast (y \ast x)) \ast z \in I \) and \( z \in I \). An ideal I of a BCK-algebra G is called closed if for all \( x \in G \), \( 0 \ast x \in I \). A subset \( Y \) of G is called a subalgebra of G if the constant 0 of G is in \( Y \), and \((Y; \ast, 0)\) itself forms a BCK-algebra.

III. TL-SUBALGEBRAS

Definition 3.1. Let G be a BCK-algebra, an L-subset \( \mu \) of G is said to be a TL-subalgebra of G if it satisfies:
(1) \( \mu(0) = 1 \);
(2) \( \mu(x \ast y) \geq \mu(x) \ast \mu(y) \) for all \( x, y \in G \).

In particular, a TL-subalgebra is simply called a L-subalgebra when \( T = \wedge \). The set of all TL-subalgebras of G and the set of all L-subalgebras of G are denoted by the symbols TL(G) and L(G), respectively.

Obviously, S is subalgebra of G if and only if characteristic function \( \chi_S \) of S is TL-subalgebra of G.

Proposition 3.2. Let G be a BCK-algebra, \( \mu \in L^G \). Then a necessary and sufficient condition for \( \mu \in L(G) \) is that every \( \mu_t(t \in L) \) is a subalgebra of G.

Proof. Let \( \mu \in L(G) \). Since \( \mu(0) = 1 \geq t \), we obtain \( 0 \in \mu_t \). If \( x, y \in A_t \), then \( \mu(x) \geq t \) and \( \mu(y) \geq t \). It follows from Definition 3.1(2) that...
\[ \mu(x \land y) \geq \mu(x) \land \mu(y) \geq t \land t = t. \]

Hence \( x \land y \in \mu \) and \( \mu \) is a subalgebra of \( G \).

On the other hand, for all \( t \in L \), \( \mu(t) \) is subalgebra of \( G \). Since \( 0 \in \mu \), we obtain \( \mu(0) \geq t \) for all \( t \in L \). Hence \( \mu(0) = 1 \). For any \( x, y \in X \), we put
\[ z = \mu(x) \land \mu(y), \]
then \( x, y \in \mu \) and hence \( x \land y \in \mu \). Thus
\[ \mu(x \land y) \geq z = \mu(x) \land \mu(y). \]
Therefore \( \mu \in L(G) \).

**Proposition 3.3.** Let \( G \) be a BCK-algebra, \( \mu \in TL(G), i \in I \), where \( I \) is any nonempty index set, then
\[ \land_{i \in I} \mu \in TL(G). \]

**Definition 3.4.** Let \( \mu, \nu \in L^S \). The \( T \)-product of \( \mu \) and \( \nu \) is defined by
\[ (\mu \land \nu)(x, y) = \mu(x) \land \nu(y) \quad \text{for all } x, y \in S. \]

**Proposition 3.5.** Let \( G \) be a BCK-algebra. If \( \mu, \nu \in TL(G) \), then \( \mu \land \nu \in TL(G \times G) \).

**Proof.** For any \( x = (x_1, x_2), y = (y_1, y_2) \in G \times G \), we have
\[ (\mu \land \nu)(0, 0) = \mu(0) \land \nu(0) = 1, \]
and
\[ (\mu \land \nu)(x \land y) = (\mu \land \nu)(x_1 \land y_1, x_2 \land y_2) \]
\[ = (\mu(x_1 \land y_1) \land \nu(x_2 \land y_2)) \]
\[ \geq (\mu(x_1) \land \mu(y_1)) \land (\nu(x_2) \land \nu(y_2)) \]
\[ = (\mu(x_1) \land \nu(x_2)) \land (\mu(y_1) \land \nu(y_2)) \]
\[ = (\mu \land \nu)(x, y) \land (\mu \land \nu)(x, y). \]
Hence \( \mu \land \nu \in TL(G \times G) \).

**Definition 3.6.** Let \( f \) denote a mapping from \( G \) into \( Y \) and let \( \mu \in L^G \) and \( \nu \in L^L \). Two \( L \)-subsets \( \nu \in L^L \) and \( f^{-1}(\nu) \in L^G \) defined by
\[ f(\mu)(y) = \{ \mu(x) \mid x \in X, f(x) = y \} \forall y \in Y \]
and
\[ f^{-1}(\nu)(x) = \{ f(\mu) \mid \nu(x) \in G \} \forall x \in G, \]
are called the image of \( \mu \) under \( f \) and the pre-image(or inverse image) of \( \nu \) under \( f \), respectively.

\[ f(\mu)(y) = \{ \mu(x) \mid x \in X, f(x) = y \} \forall y \in Y \]
and
\[ f^{-1}(\nu)(x) = \{ f(\mu) \mid \nu(x) \in G \} \forall x \in G, \]
are called the image of \( \mu \) under \( f \) and the pre-image(or inverse image) of \( \nu \) under \( f \), respectively.

**Proposition 3.7.** Let \( f \) be a BCK-algebra homomorphism from the BCK-algebra \( G \) onto the BCK-algebra \( G' \).

(1) If \( \mu \in TL(G) \), then \( f(\mu) \in TL(G') \);

(2) If \( \nu \in TL(G') \), then \( f^{-1}(\nu) \in TL(G) \).

**IV TL-IDEALS**

**Definition 4.1.** Let \( G \) be a BCK-algebra, an \( L \)-subset \( \mu \) of \( G \) is said to be a TL-ideal of \( G \) if it satisfies
\[ (F_1) \mu(0) = 1; \]
\[ (F_2) \mu(x) \geq \mu(x \land y) \forall x, y \in G. \]

**Definition 4.2.** Let \( G \) be a BCK-algebra, an \( L \)-subset \( \mu \) of \( G \) is said to be a TL-implicative ideal of \( G \) if it satisfies \( (F_3) \) and
\[ (F_3) \mu(x) \geq \mu((x \land y) \land z) \forall x, y, z \in X. \]

In particular, a TL-ideal(implicative ideal) is simply called an \( L \)-ideal(implicative ideal) when \( T = \land \). The set of all TL-ideal(implicative ideal) of \( G \) and the set of all \( L \)-ideal(implicative ideal) of \( G \) are denoted by the symbols \( TL(G) \) (\( TL(G) \)) and \( LI(G) \) (\( LI(G) \)), respectively.

Obviously, \( G \) is an ideal(implicative ideal) of \( G \) if and only if characteristic function \( \chi \) of \( I \) is TL-ideal(implicative ideal) of \( G \).

**Proposition 4.3.** Let \( G \) be a BCK-algebra. If \( \mu \in TL(G) \), then \( \mu \) is order reversing.

**Proof.** For all \( x, y \in X \) , if \( x \leq y \) , then \( x \land y = 0 \). It follows from definition 4.1 that
\[ \mu(x) \geq \mu(y) \forall x, y \in X. \]
Therefore \( \mu \) is order reversing.

**Proposition 4.4.** Let \( \mu \in TL(G) \). If the inequality \( x \land y \leq z \) holds in \( G \) , then \( \mu(x) \geq \mu(y) \forall x, y, z \in G \).

**Corollary 4.5.** Let \( G \) be a BCK-algebra, \( \mu \in L^G \). Then \( \mu \in LI(G) \) if and only if for any \( a_1, a_2, \ldots, a_n \in G \),
\[ (\cdots (x \land a_1) \land \cdots \land a_n = 0 \]
implies
\[ \mu(x) \geq \mu(a_1) \land \mu(a_2) \land \cdots \land \mu(a_n). \]

**Proposition 4.6.** Let \( G \) be a BCK-algebra, \( \mu \in L^G \). Then a necessary and sufficient condition for \( \mu \in LI(G) \) is that every \( \mu(t) \in L \) is an ideal of \( G \).

**Proposition 4.7.** Let \( G \) be a BCK-algebra, \( \mu \in L^G \). Then a necessary and sufficient condition for \( \mu \in LI(G) \) is that every \( \mu(t) \in L \) is an implicative ideal of \( G \).

**Proof.** Let \( \mu \in LI(G) \). Since \( \mu(0) = 1 \), we obtain \( 0 \in \mu \). If \( x \land (y \land z) \in \mu \) and \( y \in \mu \), then
\[ \mu(x \land (y \land z)) \geq \mu(0) \land \mu(z) \geq z. \]
It follows from definition 4.2 that
\[ \mu(x) \geq \mu((x \land (y \land z)) \land \mu(z) \geq z. \]
Hence \( x \in \mu_i \) and \( \mu_i (t \in L) \) is an implicative ideal of \( G \).

On the other hand, for all \( t \in L \), \( \mu_i \) is an implicative ideal of \( G \). Since \( 0 \in \mu_i \), we obtain \( \mu(0) \geq t \) for all \( t \in L \). Hence \( \mu(0) = 1 \). For any \( x, y, z \in G \), we put \( t = \mu(x \ast (y \ast x)) \ast z) \wedge \mu(z) \), then \( (x \ast (y \ast x)) \ast z \in \mu_i \) and \( z \in \mu_i \). Since \( \mu_i \) is an implicative ideal of \( G \), we obtain \( x \in \mu_i \). Hence \( \mu(x) \geq t = \mu(x \ast (y \ast x)) \ast z) \wedge \mu(z) \), and \( \mu \in LI(G) \).

**Definition 4.8.** Let \( \mu \in TL(G) \). \( \mu \) is said to be a TL - closed ideal of \( G \) if for all \( x \in G \),
\[
\mu(0 \ast x) \geq \mu(x).
\]

**Proposition 4.9.** Let \( \mu \in TL(G) \). Then \( \mu \) is a TL - closed ideal of \( G \) if and only if \( \mu \in TL(\mu) \).

**Proof.** Let \( \mu \) is a TL - closed ideal of \( G \), for all \( x, y \in G \), we have
\[
\mu((x \ast y) \ast x) = \mu((x \ast x) \ast y) = (0 \ast y) \geq \mu(y)
\]
and
\[
\mu(x \ast y) \geq \mu((x \ast y) \ast x) \mu(x) \geq \mu(x) \mu(y).
\]
Hence \( \mu \in TL(G) \).

On the other hand, Let \( \mu \in TL(G) \), for all \( x \in G \), we have \( \mu(0 \ast x) \geq \mu(0) \mu(x) = \mu(x) \) Therefore \( \mu \) is a TL - closed ideal of \( G \).

**Proposition 4.10.** Let \( K \) is a subalgebra of the BCK-algebra \( G \). If \( \mu \) is a TL - closed ideal of \( G \), then \( K \wedge \mu \) is a TL - closed ideal of \( K \).

**Proposition 4.11.** Let \( G \) be a BCK-algebra. If \( \mu \in TL(G) \), then the set \( A = \{ x \in G \mid \mu(x) = 1 \} \) is an ideal of \( G \).

**Proof.** Let \( \mu \in TL(G) \). If \( x \ast y \in J \) and \( y \in J \), then \( \mu(x \ast y) = \mu(y) = 1 \). By definition 4.1 we get \( \mu(x) \geq \mu(x \ast y) \mu(y) = 1 \). Thus \( \mu(x) = 1 \) and \( x \in A \). It is clear \( 0 \in A \). Hence \( A \) is an ideal of \( G \).

**Proposition 4.12.** Let \( G \) be a BCK-algebra. If \( \mu \in TL(G) \), then \( \mu \in TL(G) \).

**Proof.** Let \( \mu \in TL(G) \). Substituting \( x \) for \( y \) in \( (F_2) \), then
\[
\mu(x) \geq \mu((x \ast (x \ast x)) \ast z) \mu(z) = \mu((x \ast 0) \ast z) \mu(z) = \mu(x \ast z) \mu(z).
\]
This shows that \( \mu \) satisfies \( (F_2) \). Combining \( (F_1), \mu \in TL(G) \).

**Proposition 4.13.** Let \( G \) be a BCK-algebra. Suppose that \( \mu \in LI(G) \). Then \( \mu \in LI(G) \) if and only if , for all \( x, y \in G \), \( \mu(x) \geq \mu((x \ast (y \ast x)) \ast 0) \wedge \mu(0) = \mu(x \ast (y \ast x)) \).

**Proof.** Let \( \mu \in LI(G) \). Then for all \( x, y \in G \), we have
\[
\mu(x) \geq \mu((x \ast (y \ast x)) \ast 0) \wedge \mu(0) = \mu(x \ast (y \ast x)) \).
\]
Conversely, Suppose \( \mu(x) \geq \mu((x \ast (y \ast x)) \ast 0) \wedge \mu(0) \) for all \( x, y \in G \). Since \( \mu \in LI(G) \), by Definition 3.1, we have
\[
\mu((x \ast (y \ast x)) \ast 0) \wedge \mu(0) = \mu((x \ast (y \ast x)) \ast 0) = \mu(x \ast (y \ast x)).
\]
Hence \( \mu(x) \geq \mu((x \ast (y \ast x)) \ast 0) \wedge \mu(0) \), and \( \mu(x) \geq \mu((x \ast (y \ast x)) \ast 0) \) for all \( x, y \in G \). Since \( \mu \in LI(G) \), by Definition 3.1, we have
\[
\mu((x \ast (y \ast x)) \ast 0) \wedge \mu(0) = \mu((x \ast (y \ast x)) \ast 0) = \mu(x \ast (y \ast x)).
\]
Thus \( \mu \) satisfies \( (F_1) \). Obviously, \( \mu \) satisfies \( (F_2) \).

Therefore, \( \mu \in LI(G) \).

**REFERENCES**