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Uncertainty Reasoning based on Gradational Lattice-Valued First-Order Logic L*vfl*

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***Abstract*—The current paper discusses the uncertainty reasoning method based on gradational lattice-valued first-order logic *L_{vfl}*. For some representative uncertainty reasoning models, some concrete methods for selecting appropriate parameters during the uncertainty reasoning process based on lattice-valued first-order logic *L_{vfl}* are proposed. Emphasis is placed on the research of the consistent of *L*-type fuzzy sets of formulas.**

***Index Terms*—Uncertainty reasoning; Lattice-valued logic; consistent**

I. INTRODUCTION

Uncertainty reasoning is one of the key points in intelligent information processing, in order to build artificially computer-based systems which make computers simulate human's intelligence. Certain reasoning is based on classical logic, the reliability of the reasoning consequence is ensured by the soundness and completeness of classical logic. From the point of view of symbolism, the confidence and rationality of uncertain reasoning should be based on non-classical logics that are the extensions classical logic [5]. The research of Pavelak [6] and Novak [7] provide some basis for uncertainty reasoning based on logic. Turunen, Hajek, and Wang etc. [8-11] also got some important research results. Since 1990, Xu and his research group have always researched on the related works. In order to deal with uncertainty of two types (fuzziness and incomparability) and in two levels (in the object itself and in the course of processing it) simultaneously, Xu proposed lattice-valued logics [13-15] based on lattice implication algebra [12], studied the uncertainty reasoning based on the gradational lattice-valued propositional logic L_{vpl} and its corresponding first-order logic L_{vfl}, and proposed the uncertainty reasoning theory and method based on L_{vpl} and L_{vfl} [2, 3, 16].

Based on the above research work, the current paper studies uncertainty reasoning based on L_{vfl}. In section 2, the basic theory of uncertainty reasoning based on L_{vfl} is introduced. In section 3, some concrete methods for selecting appropriate parameters during the uncertainty reasoning process based on L_{vfl} are proposed.

II. BASIC THEORY

In L_{vfl}, we established a logical system with a semantic system and the corresponding syntactical system compatible with each other constructed by more general semantic interpretation and more general implication operator $\rightarrow$ [1]. We refer the readers to [1] for more details.

In the following, let i= I, II, $\alpha,\beta,\tau\in L$, $X,\tilde{X},Y,\tilde{Y}\in \mathrm{F}_L(\mathrm{F}_\mathrm{f})$, and $\mathrm{I}\subseteq F_L(\mathrm{F}_\mathrm{f})$.

Definition 2.1[1] The maps are defined as follows respectively:

1).

$$C_\mathrm{I}\ .\ \mathrm{F}_L(\mathrm{F}_\mathrm{f})\rightarrow \mathrm{F}_L(\mathrm{F}_\mathrm{f}).$$

$$X\mapsto C_\mathrm{I}^X.$$

where

$$C_\mathrm{I}^X(\psi)\,\square \bigwedge_{T\in\mathrm{I}}\left(\bigwedge_{\varphi\in F_f}\left(X(\varphi)\rightarrow T(\varphi)\right)\rightarrow T(\psi)\right) \qquad (1)$$

2).

$$C^{\beta,}_{\left(C_{\mathrm{Ir}}^{\varnothing},\mathrm{R}(\alpha-i)\right)}\quad \mathrm{F}_L(\mathrm{F}_\mathrm{f})\rightarrow \mathrm{F}_L(\mathrm{F}_\mathrm{f}).$$

$$X\mapsto C^{\beta,X}_{\left(C_{\mathrm{Ir}}^{\varnothing},\mathrm{R}(\alpha-i)\right)},$$

where

$$C^{\beta,X}_{\left(C_{\mathrm{Ir}}^{\varnothing},\mathrm{R}(\alpha-i)\right)}(\psi)\,\square\,\{Y(\psi)\mid Y\supseteq\beta\otimes\left(C_{\mathrm{Ir}}^{\varnothing}\cup X\right), \\ Y\ is\ \alpha-i \text{ type closed w.r.t. } \mathrm{R}\}. \qquad (2)$$

Remark 2.1 [2] From the above definition, we know C_I is the semantic operator of L*vfl* while $C^{\beta,}_{\left(C_{\mathrm{Ir}}^{\varnothing},\mathrm{R}(\alpha-i)\right)}$ is the syntactical operator.

Definition 2.2[1] Let $X\in F_L(\mathrm{F}_\mathrm{f}),\tau\in L$. If

$$\vee\left\{C^{\beta,X}_{\left(C_{\mathrm{Ir}}^{\varnothing},\mathrm{R}(\alpha-i)\right)}(\varphi)\otimes C^{\beta,X}_{\left(C_{\mathrm{Ir}}^{\varnothing},\mathrm{R}(\alpha-i)\right)}(\varphi')\mid\varphi\in\mathrm{F}_\mathrm{f}\right\}\leq\tau \qquad (3)$$

then X is said to be τ' -i type consistent w.r.t. (α,β).

The single-input-single-output (SISO) uncertainty reasoning model:

Rule: If X, then Y,
Fact: $\tilde{X}$
—————— (4)
Conclusion: $\tilde{Y}$

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Definition 2.3[1] The rule in (4) is said to be $(\alpha,\beta,\tau,\mathrm{I})$ -*i* type representable in L$_{vfl}$, if there exist $\alpha,\beta,\tau\in L$, $\mathrm{I}\subseteq F_L(\mathrm{F}_{\mathrm{f}})$ such that *X*, *Y* are τ' -*i* type consistent w.r.t. $(\alpha,\beta,\mathrm{I})$ and $C_{\mathrm{I}}^{X}\supseteq\tau'\otimes Y$.

Definition 2.4[1] The uncertainty reasoning model (4) is called an $(\alpha,\beta,\tau,\mathrm{I})$ -*i* type regular uncertainty reasoning model, if

1). The rule in (4) is $(\alpha,\beta,\tau,\mathrm{I})$ -*i* type representable.

2). New input information $\tilde{X}$ is τ' -*i* type consistent w.r.t. $(\alpha,\beta,\mathrm{I})$.

Uncertainty reasoning consequence can be given by the following equation:

$$\tilde{Y}=C_{\mathrm{I}}^{\beta\otimes\tilde{X}}.. \tag{5}$$

where β and I are obtained from the representation of the rule in (4).

The uncertainty reasoning theory and method based on lattice-valued first-order logic *L$_{vfl}$* have not only sound semantic interpretation, but also strict syntactical proof. So it has more solid theory basis and more extensive scope of application compared with other uncertainty reasoning methods which are proposed aiming at some special concrete problem. Of course, like other uncertainty reasoning methods, this method has scope of application. Firstly, it demands the uncertainty reasoning model (4) is $(\alpha,\beta,\tau,\mathrm{I})$ -*i* type regular, then we can get the reasoning consequence $\tilde{Y}=C_{\mathrm{I}}^{\beta\otimes\tilde{X}}$. The soundness of logic system *L$_{vfl}$* is ensured by condition R is α -*i* type sound w.r.t. I . If α,β,τ , I and R satisfy $C^{\beta,\tilde{X}}_{(C^{\varnothing}_{\mathrm{I}\Gamma},\mathrm{R}(\alpha-i))}\supseteq C_{\mathrm{I}}^{\beta\otimes\tilde{X}}$, i.e. the sufficient condition of the completeness, then the reasoning consequence obtained from (5) can also be obtained by a strict formal deduction in system *L$_{vfl}$*.

Note that generally α,β,τ and I are not obtained uniquely by the representation of uncertainty rule [1]. The reasoning consequence depends on the selection of α,β,τ and I despite uncertainty rule is representable and uncertain reasoning model if regular. This is just because L$_{vfl}$ is a class of lattice-valued logical systems which are different due to different α,β,τ and I . Hence, different selections of α,β,τ and I in uncertainty reasoning mean different uncertainty reasoning in different logical systems [1]. Consequently, the reasoning consequence depends on the selection of α,β,τ and I (i.e., the corresponding logical system). It follows from L$_{vfl}$ that if there exist α,β,τ and R such that

$$C^{\beta,\tilde{X}}_{(C^{\varnothing}_{\mathrm{I}\Gamma},\mathrm{R}(\alpha-i))}\in\mathrm{I}\,.$$

then the reasoning consequence obtained from (5) can also be obtained by a strict formal deduction in system *L$_{vfl}$* [1]. This follows from the weak completeness of logic system *L$_{vfl}$*. But this is a sufficient condition.

III. UNCERTAINTY REASONING EXAMPLES

The inference rule set R determines the syntactic function of *L$_{vfl}$*. The more of its number is the stronger of the syntactic function and the semantic function of *L$_{vfl}$* is [4]. Hence the chosen of R is very important, and the chosen of it depends on practical problems.

In the following, we take [2]

$$\begin{aligned}\mathrm{R}^{*}=&\left\{\left(r_2^0,t_2^*\right),\left(r_2^*,t_2^*\right),\left(r_2^{\square},t_2^*\right)\right\}\cup\left\{\left(r_1^{\theta_0},t_1^{\theta_0}\right)\mid\theta_0\in L\right\}\\&\bigcup\left\{\left(r_1^u,t_1\right)\mid u\in U\right\}\bigcup\left\{\left(r_2^u,t_1\right)\mid u\in U\right\}\\&\bigcup\left\{\left(r_3^u,t_1\right)\mid u\in U\right\},\end{aligned}$$

where,

$r_2^0(\varphi,\varphi\to\psi)=\psi$, $t_2^*(\theta,\beta)=\theta\wedge\beta$,

$r_2^*(\varphi\to\gamma,\varphi\to\psi)=\varphi\to(\gamma\wedge\psi)$,

$r_2^{\square}(\varphi\to\psi,\psi\to\gamma)=\varphi\to\gamma$,

$r_1^{\theta_0}(\varphi)=\theta_0\to\varphi$, $t_1^{\theta_0}(\alpha)=\theta_0\to\alpha$,

$r_1^u(\varphi)=(Q_u x)\varphi$, $t_1(\alpha)=\alpha$,

$r_2^u(\varphi\to\psi)=\varphi\to(Q_u x)\psi$, x is not free in φ ,

$r_3^u(\varphi\to\psi)=(Q_u x)\varphi\to\psi$, x is not free in ψ ,

$r_4^u(Q_u x)(\varphi\otimes\psi)=(Q_u x)\varphi\otimes\psi$, x is not free in ψ .

Note that R^* should be α -i type sound w.r.t. I , so we take

$$I_i=\left\{T\mid T\text{ is }\alpha\text{-i type closed w.r.t. }\mathrm{R}^*\right\}\text{, i=I, II.}$$

$$\begin{aligned}\mathrm{I}_H f\square\Big\{&\mathrm{D}_{\mathrm{F}_f}\mid\mathrm{D}_{\mathrm{F}_f}\textit{ is}\text{ an interpretation of wffs under D},\\&\text{D is an interpretation of symbols in }\mathrm{L}_{\mathrm{vfl}}\Big\}\end{aligned}$$

In *L$_{vfl}$* , let $\alpha\in L$ and $\alpha\le\bigwedge_{\theta\in L}(\theta\vee\theta')$. Then we obtain the following results [2]:

1). $\mathrm{I}_H\subseteq I_i$, *i*=I, II;

2). I_H is α -*i* type closed w.r.t. R^* ;

3). $C^{\beta,\tilde{X}}_{(C^{\varnothing}_{\mathrm{I}\Gamma},\mathrm{R}(\alpha-i))}\in I_i$ holds for any $X\in F_L(\mathrm{F}_{\mathrm{f}})$ and $\beta\in L$, *i*=I, II;

4). R^* is α -*i* type sound w.r.t. I_i , *i*=I, II;

5). $C^{\beta,X}_{(C^{\varnothing}_{\mathrm{I}\Gamma},\mathrm{R}(\alpha-i))}=C_{\mathrm{I}}^{\beta\otimes X}$ holds for any $X\in F_L(\mathrm{F}_{\mathrm{f}})$ and $\beta\in L$, *i*=I, II;

6).
$C_{\mathrm{I}}^{\beta\otimes X}=\vee\left\{\theta\mid\text{there exist }\left(p^i,(n),X,(\varphi,\theta)-(\alpha,\beta)\right)\right\}$

holds for any $X \in F_L(\mathrm{F}_{\mathrm{f}})$ and $\beta \in L$, and $\varphi \in \mathrm{F}_{\mathrm{f}}$, *i*=I, II.

Theorem 3.1 If $I \ \bigcap I_{iH\,i} \neq \varnothing$, $\beta_X \otimes X(\varphi) \leq \varphi$, then for any $\alpha, \tau \in L$, X is τ' -*i* type consistent w.r.t. $(\alpha, \beta_X, I_{\ i})$.

Proof. Suppose $T_0 \in I \ \bigcap I_{iH\,i}$.

$$\begin{aligned}C_{\mathrm{I}}^{\beta_X \otimes X}(\psi) \square & \mathop{\wedge}_{T\in \mathrm{I}}\left(\mathop{\wedge}_{\varphi\in F_f}\left(\left(\beta_X \otimes X(\varphi)\right) \to T(\varphi)\right) \to T(\psi)\right) \\ & \leq \mathop{\wedge}_{T\in \mathrm{I}}\left(\mathop{\wedge}_{\varphi\in F_f}\left(\varphi \to T(\varphi)\right) \to T(\psi)\right) \\ & \leq \mathop{\wedge}_{\varphi\in F_f}\left(\varphi \to T_0(\varphi)\right) \to T_0(\psi) \\ & = T_0(\psi),\end{aligned}$$

$$C^{\beta,X}_{\left(C^{\varnothing}_{\mathrm{Ir}}, \mathrm{R}^*(\alpha - i)\right)}(\psi) \leq C_{\mathrm{I}}^{\beta \otimes X}(\psi) \leq T_0(\psi).$$

then

$$C^{\beta,X}_{\left(C^{\varnothing}_{\mathrm{Ir}}, \mathrm{R}^*(\alpha - i)\right)}(\psi) \otimes C^{\beta,X}_{\left(C^{\varnothing}_{\mathrm{Ir}}, \mathrm{R}^*(\alpha - i)\right)}(\psi') \leq T_0(\psi) \otimes T_0(\psi') = \tilde{O}$$

Hence X is τ' -i type consistent w.r.t. $(\alpha, \beta_X, I_{\ i})$.

Corollary 3.1 If $\beta_X \otimes X(\varphi) \leq \varphi$, then for any $\alpha, \tau \in L$, X is τ' -*i* type consistent w.r.t. $(\alpha, \beta_X, I_{ii})$.

Theorem 3.2 Let $X \in F_L(\mathrm{F}_{\mathrm{f}})$ be defined as

$$X(\varphi) = \begin{cases} (k \otimes a) \oplus b, \varphi = a \in L, \\ \theta, \quad others. \end{cases}$$

Take $\alpha \leq \wedge_{\theta \in L}(\theta \vee \theta')$, $\beta_X = \theta' \wedge b'$, then X is I-*i* type consistent w.r.t. $(\alpha, \beta_X, I_{ii})$, *i*=I, II.

Theorem 3.3 Let $X \in F_L(\mathrm{F}_{\mathrm{f}})$ be defined as

$$X(\varphi) = \begin{cases} b \to a, \varphi = a \in L, \\ \theta, \quad others. \end{cases}$$

Take $\alpha \leq \wedge_{\theta \in L}(\theta \vee \theta')$, $\beta_X = \theta' \wedge b'$, then X is I-*i* type consistent w.r.t. $(\alpha, \beta_X, I_{ii})$, *i*=I, II.

Theorem 3.4 [1] Let $X, Y \in F_L(\mathrm{F}_{\mathrm{f}})$, and $I \ \subseteq F_L(\mathrm{F}_{\mathrm{f}})$. If Y is τ' -i type consistent w.r.t. $(\alpha, \beta, I\ )$ and $X \subseteq Y$, then so is X.

Theorem 3.5 Let $X \in F_L(\mathrm{F}_{\mathrm{f}})$ be defined as

$$X(\varphi) = \begin{cases} (k \otimes a^n) \oplus b, \varphi = a \in L, \\ \theta, \quad others. \end{cases}$$

where $n \in N^+$. Take $\alpha \leq \wedge_{\theta \in L}(\theta \vee \theta')$, $\beta_X = \theta' \wedge b'$, then X is *I-i* type consistent w.r.t. $(\alpha, \beta_X, I_{ii})$, *i*=I, II.

Theorem 3.6 Let $X \in F_L(\mathrm{F}_{\mathrm{f}})$ be defined as

$$X(\varphi) = \begin{cases} (k_n \otimes a^n) \oplus (k_{n-1} \otimes a^{n-1}) \oplus \cdots \oplus (k_1 \otimes a^1) b, \varphi = a \in L, \\ \theta, \quad others. \end{cases}$$

where $n \in N^+$. Take $\alpha \leq \wedge_{\theta \in L}(\theta \vee \theta')$, $\beta_X = \theta' \wedge b'$, then X is *I-i* type consistent w.r.t. $(\alpha, \beta_X, I_{ii})$, *i*=I, II.

Theorem 3.7 Let $X \in F_L(\mathrm{F}_{\mathrm{f}})$ be defined as

$$X(\varphi) = \begin{cases} a_1, \varphi = \varphi_1 \in L, \\ \vdots \\ a_n, \varphi = \varphi_n \in L, \\ \theta, \quad others. \end{cases}$$

where $n \in N^+$. Take $\alpha \leq \wedge_{\theta \in L}(\theta \vee \theta')$, $\beta_X = \mathop{\wedge}\limits_{i=1}^{n}\{a_i \to \varphi_i\} \wedge \theta'$, then X is *I-i* type consistent w.r.t. $(\alpha, \beta_X, I_{ii})$, *i*=I, II.

Theorem 3.8 Let $X \in F_L(\mathrm{F}_{\mathrm{f}})$ be defined as

$$X(\varphi) = \begin{cases} (k_1 \otimes \varphi_1) \oplus b_1, \varphi = \varphi_1 \in L, \\ \vdots \\ (k_n \otimes \varphi_n) \oplus b_n, \varphi = \varphi_n \in L, \\ \theta, \quad others. \end{cases}$$

where $n \in N^+$. Take $\alpha \leq \wedge_{\theta \in L}(\theta \vee \theta')$, $\beta_X = \theta' \wedge b_1' \wedge b_2' \cdots b_n'$, then X is *I-i* type consistent w.r.t. $(\alpha, \beta_X, I_{ii})$, *i*=I, II.

Example 3.1 Let $X, \tilde{X}, Y \in F_L(\mathrm{F}_{\mathrm{f}})$ be defined as follows:

$$X(\varphi) = \begin{cases} b \to a, \varphi = a \in L, \\ O, \quad others. \end{cases}$$

$$Y(\varphi) = \begin{cases} a, \varphi = a \in L, \\ O, \quad others. \end{cases}$$

$$\tilde{X}(\varphi) = \begin{cases} b, \varphi = a \in L, \\ O, \quad others. \end{cases}$$

Take $\alpha \leq \wedge_{\theta \in L}(\theta \vee \theta')$, $\beta_X = b'$. Note that:

1). X, $\tilde{X}$ is *I-i* type consistent w.r.t. (α, b', I_{ii}), *i*=I, II.

2). $Y \subseteq X$, then $\tilde{X}, Y$ are I-i type consistent w.r.t. (α, b', I_{ii}), i=I, II.

3). $C_{\mathrm{I}}^X \supseteq X \supseteq Y$.

Consequently, we obtain the following (α, b', I_{ii}) -*i* type regular uncertainty reasoning model:

Rule: If *X*, then *Y*,
Fact: $\tilde{X}$.

So its consequence can be obtained as follows:

$$\tilde{Y}_i(\varphi) = C_{\mathrm{I}_{ii}}^{b'\otimes\tilde{X}}(\varphi) = \underset{T\in\mathrm{T}_{ii}}{\wedge}\left(T(\varphi)\right).$$

and it is *I-i* type consistent w.r.t. (α, b', I_{ii}), *i*=I, II.

In L_{vfl}, if the complete lattice implication algebra $L = ([0,1], \vee, \wedge, ', \rightarrow)$ is the Lukasiewicz implication algebra, then this is called Lukasiewicz first-order logic system, denoted as L_{uf}. In most of uncertainty reasoning method especially for fuzzy reasoning, the formula has degree of truth ranging from 0 to 1. The practice has shown that such way of representing states of affairs is more natural and more coincident with the intuition and experience. So the research of uncertainty reasoning based on L_{uf} has wide representatively.

Example 3.2 Take L_{vfl} as Lukasiewicz first-order logic L_{uf}. Let $X, \tilde{X}, Y \in F(\mathrm{F}_{\mathrm{f}})$ be defined as follows:

$$X(\varphi) = \begin{cases} 0.8\otimes a, \varphi = a \in [0,1], \\ 0.1, \quad \mathit{others}. \end{cases}$$

$$Y(\varphi) = \begin{cases} 0.8\otimes a^n, \varphi = a \in [0,1], \\ 0.1, \quad \mathit{others}. \end{cases}$$

$$\tilde{X} = 0.8\otimes \tilde{X}$$

Take $\alpha = 0.4$, $\beta_X = 0.9$. Note that:

1). X is *I-i* type consistent w.r.t. $(0.4, 0.9, I_{ii})$, *i*=I, II.

2). $\tilde{X}, Y \subseteq X$, then $\tilde{X}, Y$ are *I-i* type consistent w.r.t. $(0.4, 0.9, I_{ii})$, *i*=I, II.

3). $C_{\mathrm{I}}^{X} \supseteq X \supseteq Y$.

Consequently, we obtain the following $(0.4, 0.9, I_{ii})$-*i* type regular uncertainty reasoning model:

Rule: If *X*, then *Y*,
Fact: $\tilde{X}$.

So its consequence can be obtained as follows:

$$\begin{aligned}\tilde{Y}_i(\varphi) &= C_{\mathrm{I}_{ii}}^{0.9\otimes\tilde{X}}(\varphi) \\ &= \underset{T\in\mathrm{T}_{ii}}{\wedge}\left[\left(\underset{\theta\geq 0.5}{\wedge}\left(0.5\otimes\theta \rightarrow T(\theta)\right)\right) \rightarrow T(\varphi)\right],\end{aligned}$$

and it is 1-*i* type consistent w.r.t. $(0.4, 0.9, I_{ii})$ in L_{uf}, *i*=I, II.

IV. CONCLUSIONS

For some special uncertainty reasoning models, some concrete methods for selecting appropriate parameters during the uncertainty reasoning process based on gradational lattice-valued first-order logic L_{vfl} are given. The practical application of these methods is the further research. This aim can be realized in the following paper.

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REFERENCES

[1] Y. Xu., D. Ruan, K.Y. Qin. and J. Liu, "Lattice-Valued Logic: An Alternative Approach to teat Fuzziness and Incomparability", Germany: Springer-Verlag, 2003.

[2] S.W. Chen, "Study on Uncertainty Reasoning in Lattice-Valued First-order Logic Based on Lattice Implication Algebra", Ph.D. Thesis, Southwest Jiaotong Univ., China, 2005.

[3] S.W. Chen, Y. Xu.. Uncertainty reasoning based on lattice-valued first-order logic L_{vfl} [A]. Proc. 2004 IEEE Coference on SMC[C], 2004, pp. 2237-2242.

[4] Z.Y. Chang, Y. Xu, J.J. Lai, X.Q. Long, "A Comparison between Lattice-Valued Propositional Logic LP(X) and Gradational Lattice-Valued Propositional Logic L_{vpl}", International Conference on Intelligent Systems and Knowledge Engineering: Chengdu, pp. 1552-1556, October 15-16 2007.

[5] J. Ma, S. Chen, and Y. Xu, "Fuzzy logic from the viewpoint of machine intelligence," Fuzzy Sets and Systems, vol. 157, no. 5, pp. 628 -634, 2006.

[6] J. Pavelka "On fuzzy logic I: Many-valued rules of inference, II: Enriched residuated lattices and semantics of propositional calculi, III: Semantical completeness of some many-valued propositional calculi", Zeitschr. F. Math. Logik und Grundlagend. Math., vol. 25: 45-52, 119-134, 447-464,1979.

[7] V. Novak, I. Perfilieva, J. Mojckojr, "Mathematical Principles of Fuzzy Logic", Kluwer, 1999.

[8] E. Turunen, "Well-defined Fuzzy Sentential Logic", J. Mathematical Logic Quarterly, vol.41, pp. 236-248,1995.

[9] P. H´ajek, "Meta-Mathematics of Fuzzy Logic", Kluwer Academic Publishers-Dordrecht, 2000.

[10] G.J. Wang, "On the logic foundation of fuzzy reasoning", J. Information Sciences, vol. 117, pp. 47-88, 1999.

[11] G.J. Wang, "Formalized theory of general fuzzy reasoning" J. Information Sciences, vol. 160, pp. 251-266, 2004.

[12] Y. Xu, "Lattice implication algebras", J. Southwest Jiaotong University, vol. 28 (1), pp. 20-27, 1993 (in Chinese).

[13] Y. Xu, K.Y. Qin, J. Liu, Z.M Song, "L-valued propositional logic L_{vpl}", J. Information Sciences, vol. 114, pp. 205-235, 1999.

[14] Y. Xu, J. Liu, Z.M. Song, K.Y. Qin, "On semantics of L-valued first- order logic L_{vfl} (I)", J. General. System, vol. 29, pp. 53-79, 2000.

[15] Y. Xu, Z.M. Song, K.Y. Qin, J. Liu, "Syntax of L-valued first order logic L_{vfl}", J. Mutiple-Valued Logic, vol. 7, pp.213-257, 2001.

[16] Y. Xu, D. Ruan, J. Liu, "Approximate reasoning based on lattice-valued propositional logic L_{vpl}", J. in D. Ruan, E.E. Kerre (Eds.). Fuzzy If-Then Rules in Computational intelligence: Theory and Applications, Kluwer Academic Publishers, pp. 81-105, 2000.