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The Basis Algebra and Approximation Operator in L-Fuzzy Rough Set

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***Abstract*—In the constructive approach of rough set, there exists three fundamental definition: approximation operator, binary relation on the universe and basis algebra. In this paper, we mainly discuss the the relation between the basis algebra and approximation operators in L-fuzzy rough set.**

***Index Terms*—Basis Algebra, Approximation Operator, L-Fuzzy Rough Set, Residuated Lattice**

I. Introduction

The theory of rough sets was firstly proposed by Pawlak[9]. It is an extension of set theory for the study of intelligent systems characterized by insufficient and incomplete information. The successful application of rough set theory in a variety of problems has amply demonstrated its usefulness.

Dubois and Prade studied first the fuzzification problem of rough sets [7]. Morsi and Yakout[8] studied a set of axioms on approximation operators of fuzzy sets and defined a special family of approximation operators of fuzzy sets using the T-norms and the residuation implicators. Additionally, Radzikowska and Kerre[4] gave another general method for the fuzzification of rough sets, called (S,T)-fuzzy rough set. They defined a broad family of fuzzy rough sets, which are determined by a triangular norm and an implicator. But the discussions of fuzzy rough set in many of article are based on fuzzy set rather than L-fuzzy set. Later, Radzikowska and Kerre[10] generalized the model of fuzzy rough set to L-fuzzy rough set based on residuated lattice and discussed some basic properties of L-fuzzy approximation operators.

Based on the study of Pawlak rough set, fuzzy rough set[7], the general fuzzy rough set[4] L-fuzzy rough set[5], we can draw a conclusion that different basis algebras get to the different properties of approximation operators[6], such as the lower and upper approximation operators of fuzzy rough set are dual, but the approximation operators aren't dual in Example 1.

Example 1: Let $\mathrm{L}_{Go} = \left([0,1], \vee, \wedge, \otimes, \rightarrow_{Go}, 0, 1\right)$ be a Goguen algebra, where for every $a, b \in \mathrm{L}_{Go}$,

$$a \vee b = \max\{a,b\},$$
$$a \wedge b = \min\{a,b\},$$
$$a \otimes b = a \cdot b,$$
$$a \rightarrow_{Go} b = \begin{cases} 1; & a \le b \\ b/a; & a > b \end{cases},$$

It is clear that the Goguen algebra is a residuated lattice. If for every $a \in \mathrm{L}_{Go}$, $\neg a = a \rightarrow_{Go} 0$.

Suppose that $y_0 \in U$, and for every $\forall x, y \in U$, $R(x,y) = 0.8$. Then

$$\neg \overline{R}_{\mathrm{L}}(\neg B_{0.4,y_0}) = \neg \overline{R}_{\mathrm{L}}(\varnothing) = U$$

$$\underline{R}_{\mathrm{L}}(B_{0.4,y_0})(x) = \inf_{y\in U}(R(x,y) \rightarrow B_{0.4,y_0}(y)) = R_{\mathrm{L}}(x,y_0) \rightarrow_{Go} 0.4 = 0.5 \cdot$$

That is $\overline{R}_{\mathrm{L}}$ and $\underline{R}_{\mathrm{L}}$ aren't dual. □

In this paper, we will discuss the influence of approximation operators on basis algebra. As the example, we will build the residuated lattice, MTL-algebra and IMTL-algebra on a new algebra, which is named as the D-algebra in this paper.

II. Preliminaries

Definition 2.1: Let $\mathrm{L} = (L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ be an algebra, where $(L, \vee, \wedge, 0, 1)$ is a complete lattice, $\otimes$ and $\rightarrow$ are binary operators. If L satisfies: for every $\forall \alpha_i, \alpha, \beta \in \mathrm{L}$, $i \in I$, I is an index set,

(1) $\neg$ be a precomplement operator, $\neg a = a \rightarrow 0$.

(2) $\forall a_i, b \in \mathrm{L}$, $\bigvee_{i\in I} \alpha_i \otimes \beta = \bigvee_{i\in I} (\alpha_i \otimes \beta)$

(3) $\forall \alpha_i, \beta \in \mathrm{L}$, $\bigvee_{i\in I} \alpha_i \rightarrow \beta = \bigwedge_{i\in I} (\alpha_i \rightarrow \beta)$

then L be named as D-algebra.

Definition 2.2: [3] By a residuated lattice, we mean an algebra $\mathrm{L} = (L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$, such that

(1) $(L, \vee, \wedge, 0, 1)$ is a bound lattice with the top element 1 and the bottom element 0.

(2) $\otimes : L \times L \rightarrow L$ is a binary operator and satisfies for all $\forall a, b, c \in L$

a) $a \otimes b = b \otimes a$,

b) $a \otimes (b \otimes c) = (a \otimes b) \otimes c$,

c) $1 \otimes a = a$,

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d) $a \le b \Rightarrow a \otimes c \le b \otimes c$.

(3) $\rightarrow: L \times L \rightarrow L$ is a residuum of $\otimes$, i.e. $\rightarrow$ satisfies for all $\forall a,b,c \in L$,

$$a \otimes b \le c \Leftrightarrow a \le b \rightarrow c.$$

Definition 2.3: [1], [2] The residuated lattice $\mathrm{L} = (L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ is called an MTL-algebra iff it satisfies the following prelinery condition, for all $a,b \in \mathrm{L}_{MTL}$,

$$(a \rightarrow b) \vee (b \rightarrow a) = 1 \cdot$$

Definition 2.4: [1], [2] The MTL-algebra L_{IMTL} is called an IMTL-algebra iff it satisfies the following condition, for every $a \in \mathrm{L}_{IMTL}$,

$$a = \sim\sim a.$$

Definition 2.5: Let L be a D-algebra, U be a nonempty universe and (U,R) be L-fuzzy approximation space. For every $A \in \mathrm{F_L}(U)$, the upper approximation set $\overline{R}_{\mathrm{L}}(A)$ and the lower approximation set $\underline{R}_{\mathrm{L}}(A)$ are defined as follow.

$$\overline{R}_{\mathrm{L}}(A)(x) = \sup_{y \in U}\left(R_{\mathrm{L}}(x,y) \otimes A(y)\right),$$

$$\underline{R}_{\mathrm{L}}(A)(x) = \inf_{y \in U}\left(R_{\mathrm{L}}(x,y) \rightarrow A(y)\right).$$

III. Basis Algebra and Approximation Operator

In this section, we define two L-fuzzy sets as follow:

$$\alpha_{y_0}(x) = \begin{cases} \alpha & x = y_0 \\ 0 & x \ne y_0 \end{cases}$$

$$\hat{a}(x) \equiv a$$

where $a \in \mathrm{L}$, $x \in U$.

Proposition 3.1: Suppose L is a D-algebra, and U is a nonempty finite universe. If for every (U,R) L-fuzzy approximation space, the upper approximation operator $\overline{R}_{\mathrm{L}}$ satisfies that $\overline{R}_{\mathrm{L}}(\varnothing) = \varnothing$ then the D-algebra satisfies (r1).

(r1). For every $\alpha \in \mathrm{L}$, $\alpha \otimes 0 = 0$.

Proof. For every $x \in U$, $\overline{R}_{\mathrm{L}}(\varnothing)(x) = \sup_{y \in U}\left(R(x,y) \otimes 0\right)$

$= \sup_{y \in U} R(x,y) \otimes 0 = 0 \cdot$

By the arbitrariness of binary relation R, $a \otimes 0 = 0$, where $a \in \mathrm{L}$.□

Proposition 3.2: Suppose L is a D-algebra, and U is a nonempty finite universe. If for every (U,R) L-fuzzy approximation space, the lower approximation operator $\underline{R}_{\mathrm{L}}$ satisfies that $\underline{R}_{\mathrm{L}}(U) = U$ then the D-algebra satisfies (r1').

(r1') For every $\alpha \in \mathrm{L}$, $\alpha \rightarrow 1 = 1$.

Proof. For every $x \in U$,

$$\underline{R}_{\mathrm{L}}(U)(x) = \inf_{y \in U}\left(R(x,y) \rightarrow 1\right) = \sup_{y \in U} R(x,y) \rightarrow 1 = 1$$

By the arbitrariness of binary relation R, $\alpha \rightarrow 1 = 1$, where $\alpha \in \mathrm{L}$.

Proposition 3.3: Suppose L is a D-algebra which satisfies (r1), and U is a nonempty finite universe. If for every (U,R) L-fuzzy approximation space, the upper approximation operator $\overline{R}_{\mathrm{L}}$ satisfies that for every $A, B \in \mathrm{F_L}(U)$, $A \subseteq B \Rightarrow \overline{R}_{\mathrm{L}}(A) \subseteq \overline{R}_{\mathrm{L}}(B)$, then the D-algebra satisfies

(r2). For every $\alpha, \beta, \gamma \in \mathrm{L}$, $\alpha \le \beta \Rightarrow \gamma \otimes \alpha \le \gamma \otimes \beta$.

Proof. Let $\alpha, \beta, r \in \mathrm{L}$, and $\alpha \le \beta$. For $y_0 \in U$,

$$\overline{R}_{\mathrm{L}}(\alpha_{y_0})(x) = R_{\mathrm{L}}(x,y_0) \otimes \alpha \le R_{\mathrm{L}}(x,y_0) \otimes \beta = \overline{R}_{\mathrm{L}}(\beta_{y_0})(x)$$

By the arbitrariness of binary relation $R_{\mathrm{L}}(x,y_0)$, for every $\gamma \in \mathrm{L}$, $\alpha \le \beta \Rightarrow \gamma \otimes \alpha \le \gamma \otimes \beta$. □

Proposition 3.4: Suppose L is a D-algebra which satisfies (r1), and U is a nonempty finite universe. If for every (U,R) L-fuzzy approximation space, R_{L} is a serial binary L-fuzzy relation, and the upper approximation operator $\overline{R}_{\mathrm{L}}$ satisfies that for every $\alpha \in \mathrm{L}$, $\overline{R}_{\mathrm{L}}(\hat{\alpha}) = \hat{\alpha}$, then the D-algebra satisfies (r3).

(r3) For every $\alpha \in \mathrm{L}$, $1 \otimes \alpha = \alpha$.

Proof. For every $x \in U$,

$$\overline{R}_{\mathrm{L}}(\hat{\alpha})(x) = \sup_{y \in U}\left(R_{\mathrm{L}}(x,y) \otimes \alpha\right) = \sup_{y \in U} R_{\mathrm{L}}(x,y) \otimes \alpha = 1 \otimes \alpha = \alpha$$

That is for $\alpha \in \mathrm{L}$, $1 \otimes \alpha = \alpha$.□

Proposition 3.5: Suppose L is a D-algebra which satisfies (r1), and U is a nonempty finite universe. If for every (U,R) L-fuzzy approximation space, R_{L} is a serial binary L-fuzzy relation, and the upper approximation operator $\overline{R}_{\mathrm{L}}$ satisfies that for every $A, B \in \mathrm{F_L}(U)$,

$$\overline{R}_{\mathrm{L}}(A \otimes B) = \overline{R}_{\mathrm{L}}(B \otimes A),$$

then the D-algebra satisfies (r4).

(r4) For every $\alpha, \beta \in \mathrm{L}$, $\alpha \otimes \beta = \beta \otimes \alpha$.

Proof. Because the R is serial, for every $x \in U$, there exists y_0 such that $R(x, y_0) = 1$. Let $\alpha, \beta \in \mathrm{L}$.

$$\overline{R}_{\mathrm{L}}(\alpha_{y_0} \otimes \beta_{y_0})(x) = \sup_{y \in U}\left(R_{\mathrm{L}}(x,y) \otimes \left(\alpha_{y_0} \otimes \beta_{y_0}\right)(y)\right)$$

$$= R_{\mathrm{L}}(x,y_0) \otimes \left(\alpha \otimes \beta\right)$$

$$= \alpha \otimes \beta$$

Similarly, $\overline{R}_{\mathrm{L}}(\beta_{y_0} \otimes \alpha_{y_0})(x) = \beta \otimes \alpha$. Thus $\alpha \otimes \beta = \beta \otimes \alpha$.□

Proposition 3.6: Suppose L is a D-algebra which satisfies (r1), and U is a nonempty finite universe. If for every (U,R) L-fuzzy approximation space, R_{L} is a serial binary L-fuzzy relation, and the upper approximation operator $\overline{R}_{\mathrm{L}}$ satisfies that for every $A, B, C \in \mathrm{F_L}(U)$,

$$\overline{R}_{\mathrm{L}}\left(A \otimes \left(B \otimes C\right)\right) = \overline{R}_{\mathrm{L}}\left(\left(A \otimes B\right) \otimes C\right)$$

then the D-algebra satisfies (r4).

(r5) For every $\alpha, \beta, \gamma \in \mathrm{L}$, $\alpha \otimes (\beta \otimes \gamma) = (\alpha \otimes \beta) \otimes \gamma$.

Proof. The proof is similar as Proposition 3.6. Let $\alpha, \beta, \gamma \in \mathrm{L}$,

$$\overline{R}_{\mathrm{L}}\left(\alpha_{y_0}\otimes\left(\beta_{y_0}\otimes\gamma_{y_0}\right)\right)(x)=\alpha\otimes(\beta\otimes\gamma)$$

$$\overline{R}_{\mathrm{L}}\left(\left(\alpha_{y_0}\otimes\beta_{y_0}\right)\otimes\gamma_{y_0}\right)(x)=(\alpha\otimes\beta)\otimes\gamma$$

Thus $\alpha\otimes(\beta\otimes\gamma)=(\alpha\otimes\beta)\otimes\gamma$. □

Proposition 3.7: Suppose L is a D-algebra which satisfies (r1) and (r1'), and U is a nonempty finite universe. If for every (U,R) L-fuzzy approximation space, the lower approximation operator $\underline{R}_{\mathrm{L}}$ and upper approximation operator $\overline{R}_{\mathrm{L}}$ satisfy that for $A,B\in \mathrm{F}_{\mathrm{L}}(U)$, $\overline{R}_{\mathrm{L}}(A)\subseteq B\Leftrightarrow A\subseteq \underline{R}_{\mathrm{L}}(B)$, then the D-algebra satisfies (r6).

(r6) For every $\alpha,\beta,\gamma\in\mathrm{L}$, $\alpha\otimes\beta\le\gamma\Leftrightarrow\alpha\le\beta\to\gamma$.

Proof. For every $x,y_0\in U$, $\alpha,\beta\in\mathrm{L}$, then

$$\overline{R}_{\mathrm{L}}(\alpha_{y_0})(x)=R_{\mathrm{L}}(x,y_0)\otimes\alpha,$$

$$\underline{R}_{\mathrm{L}}(B_{\beta,y_0})(x)=R_{\mathrm{L}}(x,y_0)\to\beta.$$

As the same time,

$$\overline{R}_{\mathrm{L}}(\alpha_{y_0})\subseteq B_{\beta,y_0}\Leftrightarrow\overline{R}_{\mathrm{L}}(\alpha_{y_0})(y_0)\le B_{\beta,y_0}(y_0)$$
$$\Leftrightarrow R_L(y_0,y_0)\otimes\alpha\le\beta$$
$$\alpha_{y_0}\subseteq\underline{R}_{\mathrm{L}}(B_{\beta,y_0})\Leftrightarrow\alpha_{y_0}(y_0)\le\underline{R}_{\mathrm{L}}(B_{\beta,y_0})(y_0)$$
$$\Leftrightarrow\alpha\le R_{\mathrm{L}}(y_0,y_0)\to\beta$$

By the arbitrariness of R_{L}, thus for every $\alpha,\beta,\gamma\in\mathrm{L}$,

$$\alpha\otimes\beta\le\gamma\Leftrightarrow\alpha\le\beta\to\gamma.\square$$

By the Definition 2.2, if D-algebra satisfies (r1)-(r6) and (r1'), then the D-algebra is a residuated lattice.

[1] M. Young, *The Technical Writer's Handbook*. Mill Valley, CA: University Science, 1989.

Proposition 3.8: Suppose L is a D-algebra which satisfies (r1) and (r3), and U is a nonempty finite universe. If (U,R) is a L-fuzzy approximation space and R_{L} is reflexive, the upper approximation operator $\overline{R}_{\mathrm{L}}$ satisfies that for $y_0\in\mathrm{L}$, $\overline{R}_{\mathrm{L}}\left(\left(B_{\alpha,y_0}\to\beta_{y_0}\right)\cup\left(B_{\beta,y_0}\to\alpha_{y_0}\right)\right)=U$, then the D-algebra satisfies (r7).

(r7) For every $\alpha,\beta\in\mathrm{L}$, $(\alpha\to\beta)\vee(\beta\to\alpha)=1$.

Proof. For every $\alpha,\beta\in\mathrm{L}$, $y_0\in U$. Because R_{L} is reflexive,

$$\overline{R}_{\mathrm{L}}\left(\left(B_{\alpha,y_0}\to\beta_{y_0}\right)\cup\left(B_{\beta,y_0}\to\alpha_{y_0}\right)\right)(y_0)=R(y_0,y_0)\otimes\left((\alpha\to\beta)\vee(\beta\to\alpha)\right)$$
$$=1\otimes\left((\alpha\to\beta)\vee(\beta\to\alpha)\right)=1\cdot$$

By (r3), $(\alpha\to\beta)\vee(\beta\to\alpha)=1$. □

By Definition 2.3, if a D-algebra satisfies (r1)-(r7) and (r1'), then the D-algebra is a MTL-algebra.

Proposition 3.9: Suppose L is a D-algebra which satisfies (r1) and (r1'), and U is a nonempty finite universe. If (U,R) is a L-fuzzy approximation space and R_{L} is reflexive, the lower approximation operator $\underline{R}_{\mathrm{L}}$ and upper approximation operator $\overline{R}_{\mathrm{L}}$ satisfy that for $A\in\mathrm{F}_{\mathrm{L}}(U)$, $\overline{R}_{\mathrm{L}}(A)=\sim\underline{R}_{\mathrm{L}}(\sim A)$, then the D-algebra satisfies (r8).

(r8) For every $\alpha\in\mathrm{L}$, $\square\square\,\alpha=\alpha$.

Proof. For every $\alpha\in\mathrm{L}$, $y_0\in U$,

$$\square\,\underline{R}_{\mathrm{L}}(\square\,\alpha_{y_0})(y_0)=\square\inf_{y\in U}\left(R_{\mathrm{L}}(y_0,y)\to\square\,\alpha_{y_0}(y)\right)$$
$$=\square\left(R_{\mathrm{L}}(y_0,y_0)\to(\alpha\to 0)\right)$$
$$=\square\left(\alpha\to 0\right)=\square\square\,\alpha\cdot\square$$

By Definition 2.4, if a D-algebra satisfies (r1)-(r8) and (r1'), then the D-algebra is an IMTL-algebra.

IV. CONCLUSION

An inevitable relation exists in basis algebra and approximation operator. In the axiomatic approach, we select some properties of approximation operators and basis algebra to create the binary relation to match approximation space. In this paper, we follow the step of axiomatic approach.

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