Construction of Basis Algebra in L-fuzzy Rough Sets

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Abstract—The approximation operator play a vital role in rough set theory. About the approximation operator, usually, we discuss three fundamental elements, including the binary relation in the universe, the basis algebra and the property of approximation operators. In the constructive approach of approximation operators, the properties of the approximation operators depend on the basis algebra and the binary relation. In this paper, we introduce the basis algebra construction approach, named as the basis algebra is built with properties of approximation operators and other certain binary relations.

Index Terms—L-fuzzy rough sets, approximation operators, basis algebra, binary relation, residuated lattice

I. INTRODUCTION

Modeling uncertain information, including fuzziness, randomicity, incompleteness and uncomparativity, is one of the main research topics in knowledge representations. Most existing approaches are based on the extensions of classical set theory such as fuzzy set theory and rough set theory.

The concept of rough set [1] was originally proposed by Pawlak as a formal tool for modeling and processing the incomplete information in information systems. In the rough set theory, the core idea of rough set is to approximate the knowledge with uncertainty by using two “certain” definitions. The two “certain” definitions are named as the lower and the upper approximation sets.

The lower and the upper approximation operators are the two unary mappings in the universe. To real applications, we select a certain rough set model to approximate the uncertain information. For instance, if there is the fuzzy information, we take the fuzzy rough set into account.

Many new types of rough sets theories had been put forward, such as fuzzy rough sets, L-fuzzy rough sets and general rough sets. In the axiomatic approach, such as ref. [1,2], the approximation operators had been defined as two unary operators which satisfy some axioms in the universe. In the axiomatic approach of approximation operators it is vital to find the binary relation to construct the approximation space.[3]

In the constructive approach of rough approximation operators, many work focused on the properties of approximation operators in different binary relations. By compared with the rough sets[1], fuzzy rough sets[4-6] and L-fuzzy rough sets[7,8,9], it can be found that the properties of approximation operators based on different basis algebras (such as boolean algebra, [0,1] and lattice) are different, even in the same binary relations and the forms of approximation operators. In this paper, we discuss the L-fuzzy rough sets based on residuated lattice, IMTL algebra and Boolean algebra. The differences of these rough sets will be shown under the influence of basis algebra.

II. PRELIMINARIES

Definition 1[9]. By a residuated lattice, we mean an algebra \((L,\lor,\land,\to,0,1)\) such that

(1) \((L,\lor,\land,0,1)\) is a bound lattice with the top element 1 and the bottom element 0.
(2) \(\otimes : L \times L \to L\) is a binary operator and satisfies for \(\forall a,b,c \in L\),
   \[a \otimes b = b \otimes a, a \otimes (b \otimes c) = (a \otimes b) \otimes c,\]
   \[1 \otimes a = a, a \leq b \Rightarrow a \otimes c \leq b \otimes c.\]
(3) \(\to : L \times L \to L\) is a residuum of \(\otimes\), i.e. \(\to\) satisfies for all \(a,b,c \in L\),
   \[a \otimes b \leq c \Rightarrow a \leq b \to c.\]

A residuated lattice \((L,\lor,\land,\to,0,1)\) is called complete iff the underlying lattice \((L,\lor,\land,0,1)\) is complete. Given a residuated lattice \(L\), we define the precomplement operator \(\neg : L \to L\) as follows: \(\forall a \in L\),
   \[\neg a = a \to 0.\]

Theorem 1[9] Suppose \((L,\lor,\land,\neg,0,1)\) is a residuated lattice, and \(\neg\) is the precomplement operator on \(L\). Then \(\forall a,b,c \in L\),
(1) \(a \otimes b \leq a, a \to b \geq b.\)
(2) \(a \to (b \to c) = (a \otimes b) \to c,\)
(3) \(a \to (b \to c) = b \to (a \to c).\)
(4) \(a \leq b \Rightarrow a \to b = 1.\)
(5) \(a \neg \neg a, \neg \neg a = a.\)
(6) \(a \to b = b \to a = (a \otimes b).\)
(7) \(a \to b \leq (a \otimes b), a \otimes b \leq (a \to b).\)
(8) If \(L\) is a complete lattice, then
   \[\left(\bigvee_{\forall a} b = \bigvee_{\forall a} (a \otimes b), a \to \left(\bigvee_{\forall a} b\right) = \left(\bigvee_{\forall a} (a \to b)\right).\]

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\[
\begin{align*}
\left( \bigvee_{i=1}^d a_i \right) & \to b = \bigwedge_{i=1}^d (a_i \to b), \\
a & \to \left( \bigvee_{i=1}^d b_i \right) \geq \bigwedge_{i=1}^d (a \to b_i), \\
\left( \bigwedge_{i=1}^d a_i \right) & \to b \geq \bigvee_{i=1}^d (a_i \to b).
\end{align*}
\]

**Definition 2**[9] The residuated lattice \((L, \lor, \land, \otimes, \to, 0, 1)\) is called an MTL-algebra iff it satisfies the following preliminary condition, for \(\forall a, b \in L\), \((a \to b) \lor (b \to a) = 1\).

**Definition 3**[11] The MTL-algebra \(L_{\text{MTL}}\) is called an IMTL-algebra iff it satisfies the following condition: for \(\forall a \in L_{\text{MTL}}, \sim a = a\).

**Theorem 3**[11] Suppose \((L, \lor, \land, \otimes, \to, 0, 1)\) is an MTL algebra, and \(\sim\) is the precomplement operator on \(L\). Then for \(\forall a, b \in L\),

1. \((a \to b) \lor (b \to a) = 1\).
2. \((a \to b) = (a \otimes b) \lor (a \to b)\).

**Definition 5**[7] Suppose \(U\) is a non-empty finite universe and \(R_L\) is an L-fuzzy binary relation on \(U\) based on residuated lattice. \((U, R_L)\) is called an L-fuzzy approximation space based on the residuated lattice. For any set \(A \subseteq F_L(U)\), the lower and upper approximation \(R_L(A)\) and \(\overline{R}_L(A)\) of \(A\) with respect to the approximation space \((U, R_L)\) are L-fuzzy sets on \(U\) whose membership functions are respectively defined by

\[
\begin{align*}
R_L(A)(x) &= \inf_{y \in L} (R_L(x, y) \to A(y)) \\
\overline{R}_L(A)(x) &= \sup_{y \in L} (R_L(x, y) \otimes A(y)).
\end{align*}
\]

The pair \((R_L(A), \overline{R}_L(A))\) is referred to as an L-fuzzy rough set. \(R_L, \overline{R}_L: F_L(U) \to F_L(U)\) are referred to as lower and upper L-fuzzy approximation operators.

Radzikowska et al. have proved some properties of L-fuzzy rough approximation operations based on residuated lattice in [5]. These properties of L-fuzzy rough sets have been collected in Theorem 3, the others as the supplement of [5] are in Theorem 4[3].

**Theorem 4**[5] Let \((L, \lor, \land, \otimes, \to, 0, 1)\) be a complete residuated lattice, \((U, R_L)\) is L-fuzzy approximation space. Then for \(\forall A, B \in F_L(U)\),

1. \(R_L(U) = U, \overline{R}_L(\emptyset) = \emptyset\).
2. If \(A \subseteq B\), then \(R_L(A) \subseteq R_L(B), \overline{R}_L(A) \subseteq \overline{R}_L(B)\).
3. \(R_L(A \cup B) = R_L(A) \cap R_L(B), \overline{R}_L(A \cup B) = \overline{R}_L(A) \cup \overline{R}_L(B)\).
4. \(R_L(A \cup B) \supseteq R_L(A \cup \overline{R}_L(B), \overline{R}_L(A \cap B) \subseteq \overline{R}_L(A) \cap \overline{R}_L(B)\).

**Theorem 5**[3,12] Let \((L, \lor, \land, \otimes, \to, 0, 1)\) be a complete residuated lattice, \((U, R_L)\) is L-fuzzy approximation space. Then for \(\forall A, B \in F_L(U)\),

1. \(R_L(A) \subseteq \overline{R}_L(A), \overline{R}_L(A) \subseteq \overline{R}_L(A)\).
2. \(\overline{R}_L(A) = R_L(A) = \overline{R}_L(A) = \overline{R}_L(A)\).

**Theorem 6**[3,12] Let \((L, \lor, \land, \otimes, \to, 0, 1)\) be a complete IMTL-algebra, \((U, R_L)\) is L-fuzzy approximation space. Then for \(\forall A, B \in F_L(U)\),

1. \(R_L(A) = R_L(A) = \overline{R}_L(A) = \overline{R}_L(A)\).

In Theorem 6, we just list the properties which are different from the the residuated lattice-fuzzy rough set. The major difference between the L-fuzzy rough sets based on residuated lattice and IMTL-algebra is the duality of the lower and upper approximation operators. The duality is important for the axiomatic approach. Usually, in the axiomatic approach, the relation is defined by the upper approximation operator. Through the duality (or the semi-duality) of the approximation operator, the relation is loaded in the lower approximation operator. Without the duality, the process add some other conditions to load the relation by the lower approximation operator, such as the condition \(\sim a \mid a \in L = L^\sim\).

**III. THE BASIS ALGEBRA IN L-FUZZY ROUGH SETS**

In an IMTL algebra-fuzzy rough set, we prove the lower and the upper approximation operators are dual, but it does not hold in a residuated lattice-fuzzy rough set. One question arises: Which property in the IMTL-algebra is virtual for the duality of approximation operators? Another question is: If we want the approximation operators to satisfy some properties, such as the duality in the axiomatic approach, which properties should the corresponding basis algebra satisfy?

**Definition 6** Let \((L, \lor, \land, \otimes, \to, 0, 1)\) be an algebra, where \((L, \lor, \land, \otimes, 0, 1)\) is a complete lattice. \(\otimes, \to\): \(L \times L \to L\) are the binary operators. If \(L\) satisfies:

1. \(\sim: L \to L\) is a unary operator. For all \(a \in L\), \(\sim a = a \to 0\);
2. \(\forall \alpha, \beta \in L, \otimes\alpha, \beta = \land(\alpha, \land \beta), \to \alpha, \beta = \lor(\alpha, \lor \beta)\).
3. \(\forall \alpha, \beta \in L, \to \alpha, \beta = \land(\alpha, \land \beta)\)

Then the \(L\) is named as D-algebra.

By selecting the certain L-fuzzy sets, such as \(B_{x,y}\), we can prove all of the following theorems, where for all \(a \in L, x \in U\)

\[
\begin{align*}
\alpha_{x,y}(x) &= \begin{cases} 
\alpha & x = y, \\
0 & x \neq y
\end{cases}, \quad B_{x,y}(x) &= \begin{cases} 
1 & x = y, \\
0 & x \neq y
\end{cases}.
\end{align*}
\]

**Theorem 7** Let \(L\) be a D-algebra and \(U\) be a non-empty universe. If for all L-fuzzy approximation spaces
(U, R), the upper approximation operators \( \overline{R}_c \) satisfy \( \overline{R}_c(\emptyset) = \emptyset \), then

(r1) for all \( a \in L \), \( a \otimes 0 = 0 \).

**Theorem 8** Let \( L \) be a D-algebra and \( U \) be a non-empty universe. If for all L-fuzzy approximation spaces \((U, R)\), the upper approximation operators \( \overline{R}_c \) satisfy \( \overline{R}_c(U) = U \), then

(r1') for all \( a \in L \), \( a \rightarrow 1 = 1 \).

**Theorem 9** Let \( L \) be a D-algebra and \( U \) be a non-empty universe. If for all L-fuzzy approximation spaces \((U, R)\), the upper approximation operators \( \overline{R}_c \) satisfy for all \( A, B \in F_c(U) \),

\[
A \subseteq B \Rightarrow \overline{R}_c(A) \subseteq \overline{R}_c(B),
\]

then

(r2) for all \( a, b, c \in L \), \( a \leq b \Rightarrow c \otimes a \leq c \otimes b \).

**Theorem 10** Let \( L \) be a D-algebra which satisfies (r1) and \( U \) be a non-empty universe. If \((U, R)\) is an L-fuzzy approximation space, the L-fuzzy binary relation \( R \) is reflexive, and the upper approximation operators \( \overline{R}_c \) satisfy for all \( a \in L \), \( \overline{R}_c(\hat{a}) = \hat{a} \), then

(r3) for all \( a \in L \), \( 1 \otimes a = a \).

**Theorem 11** Let \( L \) be a D-algebra which satisfies (r1) and (r3), and \( U \) be a non-empty universe. If \((U, R)\) is an L-fuzzy approximation space, the L-fuzzy binary relation \( R \) is reflexive, and the upper approximation operators \( \overline{R}_c \) satisfy for all \( A, B \in F_c(U) \),

\[
\overline{R}_c(A \otimes B) = \overline{R}_c(B \otimes A),
\]

then

(r4) for all \( a, b \in L \), \( a \otimes b = b \otimes a \).

**Theorem 12** Let \( L \) be a D-algebra which satisfies (r1) and (r3), and \( U \) be a non-empty universe. If \((U, R)\) is an L-fuzzy approximation space, the L-fuzzy binary relation \( R \) is reflexive, and the upper approximation operators \( \overline{R}_c \) satisfy for all \( A, B, C \in F_c(U) \),

\[
\overline{R}_c((A \otimes B) \otimes C) = \overline{R}_c(A \otimes (B \otimes C))
\]

then

(r5) for all \( a, b, c \in L \), \( a \otimes (b \otimes c) = (a \otimes b) \otimes c \).

**Theorem 13** Let \( L \) be a D-algebra which satisfies (r1) and (r1'), and \( U \) be a non-empty universe. If for all L-fuzzy approximation spaces \((U, R)\), the approximation operators \( \overline{R}_c, \overline{R}_b \) satisfy for all \( A, B \in F_c(U) \),

\[
\overline{R}_c(A) \subseteq B \Leftrightarrow A \subseteq \overline{R}_b(B),
\]

then

(r6) for all \( a, b, c \in L \), \( a \otimes b \leq c \Leftrightarrow a \leq b \rightarrow c \).

By the definition of the residuated lattice, if the D-algebra satisfies (r1)-(r6) and (r1'), then the D-algebra is a residuated lattice. Following these steps, we can construct the MTL-algebra and the IMTL-algebra.

**Theorem 14** Let \( L \) be a D-algebra which satisfies (r1) and (r3), and \( U \) be a non-empty universe. If \((U, R)\) is an L-fuzzy approximation space, the L-fuzzy binary relation \( R \) is reflexive, and the upper approximation operators \( \overline{R}_c \) satisfy for all \( A, B \in F_c(U) \),

\[
\overline{R}_c \left( \left( B_{a, b} \rightarrow b \right) \cup \left( B_{b, a} \rightarrow a \right) \right) = U
\]

then

(r7) for all \( a, b, c \in L \), \( (a \rightarrow b) \vee (b \rightarrow a) = 1 \).

**Theorem 15** Let \( L \) be a D-algebra which satisfies (r1) and (r1'), and \( U \) be a non-empty universe. If for all L-fuzzy approximation spaces \((U, R)\), the approximation operators \( \overline{R}_c, \overline{R}_b \) satisfy for all \( A \in F_c(U) \),

\[
\overline{R}_c(\sim a) = \overline{R}_b(\sim a)
\]

then

(r8) for all \( a \in L \), \( \sim a = a \).

By the definition of the IMTL-algebra, if the D-algebra satisfies (r1)-(r8) and (r1'), then these D-algebra is an IMTL-algebra.

In this section, as the sufficient conditions, the properties of the approximation operators are the special ones.

**IV. CONCLUSIONS**

The choosing process of the basis algebra is similar to the axiomatic process of the approximation operators. The axiomatic approach is the process that finds the binary relation based on the axiom set and the basis algebra. The basis algebra choosing approach is the process that constructs the basis algebra with the properties of the approximation operator and the binary relation. As the axiomatic approach[3,6,10], in the basis algebra choosing approach, the condition set of the basis algebra choosing approach is not unique, such as the fuzzy rough set (r1)-(r8), (r1') aren't our only choice. We can find other conditions to replace them. For example, the condition “(r3)” For every binary relation \( R \), \( \overline{R}_c(U) = U \) can replace (r3).

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