Reproducing Kernel Functions Represented by Form of Polynomials

Sen Zhang, Lei Liu, and Luhong Diao
College of Applied Sciences, Beijing University of Technology, Beijing, China
Email: zhangsen@yahoo.com

Abstract—By re-defining the inner product of a reproducing kernel space, the reproducing kernel functions of that space can be represented by form of polynomials without changing any other conditions, and the higher order of the derivatives, the simpler of the reproducing kernel function expressions. Such expressions of reproducing kernel functions are the simplest from the computational point of view, resulting in speed and accuracy significant improvement in scientific and engineering applications. The performance of such reproducing kernel functions is shown to be very encouraging by experimental results.

Index Terms—Reproducing Kernel Space; Reproducing Kernel Function; Hilbert Space

I. INTRODUCTION

Bergman [1−7], J. Mercer [8], E. H. Moore and S. Bochner respectively proposed a special function in the 1920s in their different research fields, i.e., Bergman called $EI(z, Z, t)$ as the producing function of differential equations (now it is also called Bergman kernel function), J. Mercer named $K(x, y)$ as positive definite kernel, etc.

In the 1950s, N. Aronszajn [9] summarized the previous related research results and used “reproducing kernel function” as the identical term for these different functions, so the foundations of reproducing kernel theory was set up. Since then, many researchers have contributed much work to the development and improvement of reproducing kernel theory.

In 1986, Cui [10] proved that $W_2^1[a, b]$ is a Hilbert space with reproducing kernel function, and exactly expressed the reproducing kernel function of $W_2^1[a, b]$ by finite terms. Hence, the application of reproducing kernel theory began in many areas. In recent years, many research reports showed that some problems could be solved in reproducing kernel space $W_2^m[a, b]$ because the reproducing kernel function of $W_2^m[a, b]$ could be exactly expressed by finite terms. For instance, the problem of infinite linear equations, the singular boundary problem, the period problem, the nonlinear problem and the inverse problem of wave equation could be solved in $W_2^m[a, b]$ with satisfied solutions [11-19].

However, the expression of the reproducing kernel function of $W_2^m[a, b]$ used in previous papers is too complicate, and the complexity of the expression of the reproducing kernel function of $W_2^m[a, b]$ will increase while $m$ becomes larger, which will lead to some significant difficulties in computation. For example, the reproducing kernel function of $W_2^7[0, 1]$ is a segmental function expressed by $x^3$, $e^{2x}$, $\sin \alpha x$, $\cos \beta x$ and some fundamental operations (such as add, subtract, multiply and divide, etc), and could be full of more than 9 or 10 A4 pages if printed out. Moreover, the higher of differential orders, the more complicate of the reproducing kernel function. Therefore, the complicate reproducing kernel function could lead to some serious problems in computation, and some problems may become very difficult if the function in these problems with high smoothness degrees.

Based on the reproducing kernel space $W_2^m[a, b]$ established by Professor Cui Minggen and by re-defining the inner product, the reproducing kernel functions of $W_2^m[a, b]$ can be significantly simplified and expressed by polynomials without changing any other conditions. In this case, the reproducing kernel functions could be represented by piecewise polynomials, and the higher order of derivatives, the simpler of the reproducing kernel function expressions. Such expressions of reproducing kernel functions are the simplest from the computational viewpoint, the speed and accuracy could be significantly improved in scientific and engineering applications. The performance of such reproducing kernel functions is shown to be very encouraging by experimental results.

II. RPKS $W_2^m[a, b]$

The function space $W_2^m[a, b]$ is defined as follows:

$W_2^m[a, b] = \{ f(x) | f^{(m−1)}(x) \text{ is absolutely continuous, } \int_{a}^{b} |f^{(m)}(x)|^{2} dx < \infty, x \in [a, b] \}$

The inner product and the norm in the function space $W_2^m[a, b]$ are defined as follows respectively:

for any functions $f(x), g(x) \in W_2^m[a, b]$,

\[ <f, g> = \sum_{i=0}^{-m-1} f^{(i)}(a)g^{(i)}(a) + \int_{a}^{b} f^{(m)}(x)g^{(m)}(x) dx, \]  

\[ \|f\| = \sqrt{<f, f>}. \]
It is easy to prove that $W_2^n[a,b]$ is an inner space with the definitions of (2.2).

**Theorem 2.1** Function space $W_2^n[a,b]$ is a Hilbert Space.

*Proof:* Suppose $f_n(x)$ ($n = 1, 2, \ldots$) is a Cauchy sequence in $W_2^n[a,b]$, i.e., if $n \to \infty$, then

$$\left| f_{m+p}(x) - f_n(x) \right|^2 \to 0$$

Therefore, we have $f_{m+p}(x) - f_n(x) \to 0$, $i = 0, 1, \ldots, m - 1$, and

$$\int_a^b \left| f_{m+p}(x) - f_n(x) \right|^2 dx \to 0.$$  (2.4)

which indicates that for any $i$ ($0 \leq i \leq m - 1$), the sequence $f_{n,i}(x)$ ($n = 1, 2, \ldots$) is a Cauchy sequence and $f_{n,i}(x)$ ($n = 1, 2, \ldots$) is a Cauchy sequence in space $L^2[a,b]$. So, there exist unique real number $\lambda_i$ ($i = 0, 1, \ldots, m - 1$) and unique function $h(x) \in L^2[a,b]$, satisfy the following:

$$\lim_{n \to \infty} f_{n,i}(x) = \lambda_i \quad (0 \leq i \leq m - 1)$$

and

$$\lim_{n \to \infty} \int_a^b \left| f_n(x) - h(x) \right|^2 dx = 0.$$

Suppose

$$g(x) = \sum_{k=0}^{m-1} \lambda_k (x-a)^k + \int_a^x \int \cdots \int_a^x h(x)(dx)^m,$$  (2.6)

since $h(x) \in L^2[a,b]$ , hence $g^{(m-1)}(x) = \lambda_{m-1} \int_a^x h(x)dx$ is absolutely continuous in $[a,b]$ , and $g^{(m)}(x) = h(x)$ is true almost everywhere in $[a,b]$. So, $g(x) \in W_2^n[a,b]$ and $g^{(i)}(x) = \lambda_i (0 \leq i \leq m - 1)$. Moreover, we have:

$$\left\| g(x) - f_n(x) \right\|^2 = \sum_{i=0}^{m-1} \left| \lambda_i - \lambda_i \right|^2 + \int_a^b \left| f_n^{(m)}(x) - h(x) \right|^2 dx \to 0.$$  (2.7)

Hence, function space $W_2^n[a,b]$ is a Hilbert Space.

**Lemma 2.2** $W_2^n[a,b]$ is a reproducing kernel space if and only if for any $x \in [a,b]$ , $I : f \to f(x)$ is a bounded functional in $W_2^n[a,b]$ .

**Theorem 2.3** Function space $W_2^n[a,b]$ is a reproducing kernel space.

*Proof:* In fact, suppose $x \in [a,b]$ and $f(x) \in W_2^n[a,b]$ , we have

$$f^{(1)}(x) = f^{(1)}(a) + \int_a^x f^{(1)}(x)dx,$$

and

$$\left| f^{(1)}(x) \right| \leq \int_a^b \left| f^{(1)}(x) \right| dx \leq M_{\|f\|_{W_2^n}}.$$

Therefore

$$\| f^{(1)}(x) \| \leq M_{\|f\|_{W_2^n}}.$$

Note that for any $i$ ($0 \leq i \leq m - 1$), we have

$$\left| f^{(i)}(x) \right| \leq \left| f^{(i)}(a) \right| + \int_a^b \left| f^{(i)}(x) \right| dx \leq M_{\|f\|_{W_2^n}}.$$

Therefore

$$\| f^{(1)}(x) \| \leq M_{\|f\|_{W_2^n}}.$$

Similarly we have

$$\| f^{(m-2)}(x) \| \leq M_{\|f\|_{W_2^n}}.$$

So, $I$ is bounded functional in $W_2^n[a,b]$ and $W_2^n[a,b]$ is a reproducing kernel space.

Now, let’s find out the expression form of the reproducing kernel function $R_n(x, y)$ in $W_2^n[a,b]$ .

Suppose $R_n(x, y)$ is the reproducing kernel function of $W_2^n[a,b]$ , then for any fixed $y \in [a,b]$ and any $f(x) \in W_2^n[a,b]$ , $R_n(x, y)$ must satisfy the following:
< f(x), R_n(x, y) >= f(y) \quad (2.12)

Based on (2.2), we have:

\[
< f(x), R_n(x, y) >= \sum_{j=0}^{n-1} f^{(j)}(a) \frac{\partial R_n(a, y)}{\partial x} + \int_a^b \frac{\partial R_n(x, y)}{\partial x} \, dx \quad (2.13)
\]

and

\[
\int_a^b \frac{\partial R_n(x, y)}{\partial x} \, dx = \sum_{j=0}^{n-1} (-1)^j f^{(j)}(a) \frac{\partial R_n(a, y)}{\partial x} + \int_a^b \frac{\partial R_n(x, y)}{\partial x} \, dx \quad (2.14)
\]

by variable substitution, we have

\[
\sum_{j=0}^{n-1} (-1)^j f^{(j)}(x) \frac{\partial R_n(x, y)}{\partial x} = \sum_{j=0}^{n-1} (-1)^{m-j} f^{(j)}(x) \frac{\partial R_n(x, y)}{\partial x} \quad (2.15)
\]

Moreover,

\[
< f(x), R_n(x, y) >= \sum_{j=0}^{n-1} f^{(j)}(a) \frac{\partial R_n(a, y)}{\partial x} - \sum_{j=0}^{n-1} (-1)^{m-j} f^{(j)}(a) \frac{\partial R_n(a, y)}{\partial x} \quad (2.16)
\]

and

\[
\frac{\partial^2 R_n(x, y)}{\partial x^2} = \delta(x-y), \quad (2.17)
\]

We know that equation (2.14) has characteristic equation \( \lambda^{2m} = 0 \), and the eigenvalue \( \lambda = 0 \) is a root whose multiplicity is \( 2m \). Therefore, the general solution of equation (2.14) is as follows:

\[
R_n(x, y) = \sum_{j=1}^{2m} c_j(y) x^{j-1}, \quad x < y, \\
R_n(x, y) = \sum_{j=1}^{2m} d_j(y) x^{j-1}, \quad x > y. \quad (2.18)
\]

Now we are ready to calculate the coefficients \( c_j(y) \) and \( d_j(y) \), \( i = 1, \ldots, 2m \).

Since

\[
(1)^m \frac{\partial^2 R_n(x, y)}{\partial x^{2m}} = \delta(x-y). \quad (2.19)
\]

Then we have:

\[
\frac{\partial^i R_n(x, y)}{\partial x^i} = \frac{\partial^i}{\partial x^i} \frac{\partial R_n(y, y)}{\partial x} \quad i = 0, 1, \ldots, 2m - 2 \quad (2.20)
\]

and

\[
(1)^m \left( \frac{\partial^{2m-1} R_n(y+, y)}{\partial x^{2m-1}} - \frac{\partial^{2m-1} R_n(y-, y)}{\partial x^{2m-1}} \right) = 1 \quad (2.21)
\]

The above equations in (2.17) and (2.18) provided 2m conditions for solving the coefficients \( c_j(y) \) and \( d_j(y) \) \( i = 1, \ldots, 2m \) in equation (2.16). Note that equation (2.15) provided 2m boundary conditions, so we have 4m equations, i.e., (2.15), (2.17) and (2.18). It is easy to know these 4m equations are linear equations with the variables \( c_j(y) \) and \( d_j(y) \), and the \( c_j(y) \) and \( d_j(y) \) could be calculated out by many methods. As long as the coefficients \( c_j(y) \) and \( d_j(y) \) are known, the exact expression of the producing kernel function \( R_n(x, y) \) of \( W_2^{2m}[a, b] \) could be calculated out from equation (2.16). The expression of \( R_n(x, y) \) is a piecewise polynomial with 2m − 1 degrees.

Note: if the functions in space \( W_2^{2m}[a, b] \) require more special boundary conditions, e.g., the boundary conditions of the second order differential equations as follows:

\[
u(a) = \alpha, \nu(b) = \beta, \quad \text{or} \quad \nu(a) = \alpha, \nu'(a) = \beta \quad \text{or} \quad \text{the linear boundary conditions:} \]

\[a_1 \nu(a) + b_1 \nu'(a) = \alpha, \quad a_2 \nu(b) + b_2 \nu'(b) = \beta \quad \text{or} \quad \text{the periodic linear boundary conditions:} \]

\[\nu(a) = \nu(b), \quad \nu'(a) = \nu'(b)\]
These different kinds of boundary conditions could be contained in space $W^m_{\omega}[a,b]$ after homogenization, i.e., we can find the reproducing kernel function $R_m(x,y)$ which satisfies these boundary conditions. In all, the reproducing kernel space $W^m_{\omega}[a,b]$ has the simplest reproducing kernel function $R_m(x,y)$ represented by polynomials and could be applied in many areas.

III. RPK FUNCTIONS OF $W^m_2[0,1]$

We are ready to present some expressions of reproducing kernel function in $W^m_2[0,1]$ by using the approaches proposed in the above sections.

A. RPK function $R_1(x,y)$ in $W^1_2[0,1]$

$$R_1(x,y) = \begin{cases} 1 + x, & x \leq y, \\ 1 + y, & x > y, \quad x,y \in [0,1] \end{cases}$$

The 3-d and 2-d images of the reproducing kernel function $R_1(x,y)$ are shown in the following Fig. 3.1 and Fig. 3.2 respectively. In Fig. 3.1, $x,y \in [0,1]$, and Fig. 3.2, $x \in [0,1]$, $y$ takes values 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.9, respectively.

B. RPK function $R_2(x,y)$ in $W^2_2[0,1]$

$$R_2(x,y) = \begin{cases} \frac{1}{6} - \frac{1}{2}x + \frac{1}{2}y(2+x), & x \leq y, \\ \frac{1}{6} - \frac{1}{2}x + \frac{1}{2}y(2+y), & x > y, \quad x,y \in [0,1]. \end{cases}$$

C. RPK function $R_3(x,y)$ in $W^3_2[0,1]$

$$R_3(x,y) = \begin{cases} \frac{1}{120} + \frac{1}{2}y(x^2 + y^2)(3+x) + x(1 - x)^2, & x \leq y, \\ \frac{1}{120} + \frac{1}{2}y(x^2 + y^2)(3+y) + y(1 - y)^2, & x > y, \quad x,y \in [0,1] \end{cases}$$

D. RPK function $R_7(x,y)$ in $W^7_2[0,1]$

$$R_7(x,y) = \begin{cases} \frac{1}{627020800} + \frac{1}{576}(x+y)^2 + \frac{2864}{604800} - x^2 y^2, & x \leq y, \\ \frac{1}{627020800} + \frac{1}{576}(x+y)^2 + \frac{2864}{604800} - x^2 y^2, & x > y, \quad x,y \in [0,1] \end{cases}$$

E. Numerical Examples

Some kinds of differential and integral equations with boundary or initial conditions have been studied and solved in literature [11-19] by using the properties of reproducing kernel functions, and the approximate solutions $u_n$ where $n$ indicates the number of computing nodes have been obtained. To test the performance of the reproducing kernel functions.
presented in this paper, let’s consider the following differential equation with boundary conditions:
\[
\begin{cases}
u'' + u' + u = f(x), & 0 \leq x \leq 1 \\
u(0) = 0, u'(0) = 0.
\end{cases}
\] (4.1)

Let \(u_n\) be the approximate solution, \(u\) the exact solution, and \(\varepsilon_n = \max\{|u_n - u|\}_{x \in [0,1]}\) indicate the absolute error. To solve the equation (4.1), a program was developed by using Mathematica 5.1 and run on a PC with CPU 2.0 GHz and RAM 512Mb. The experimental results are as follows:

(1) take the reproducing kernel function proposed in [11-19] and let \(n = 100\), set \(\varepsilon_n \leq 0.00005\), the running time is 759.547s.

(2) take the reproducing kernel function proposed in this paper and let \(n = 100\), set \(\varepsilon_n \leq 0.000014\), the running time is 7.204s.

(3) take the reproducing kernel function proposed in this paper and let \(n = 400\), set \(\varepsilon_n \leq 8 \times 10^{-7}\), the running time is 148.61s.

The experimental results showed that the reproducing kernel function proposed in this paper could improve the computational speed significantly (more than 100 times). Moreover, we have proved that if \(n \to \infty\), then \(\varepsilon_n \to 0\). Thus, the computational accuracy could be improved too.

IV. CONCLUSIONS

It is well known that the Hilbert Space theory is the foundation of modern mathematics. The reproducing kernel space \(W_2^n[a,b]\) is just a Hilbert Space with some special properties. So, \(W_2^n[a,b]\) can inherit all the properties of the Hilbert Space and possess some special and better properties, which could make some problems be solved easier. For example, \(L^2[a,b]\) is a complete Hilbert space. Many problems studied in \(L^2[a,b]\) requires large amount of integral computations, and such computations may be very difficult in some cases. Thus, the numerical integrals have to be calculated in the cost of losing some accuracy. However, the properties of the producing kernel space \(W_2^n[a,b]\) (see (2.12)) require no more integral computation for some functions, instead of computing some values of a function at some nodes. This simplification of integral computation not only improves the computational speed, but also improves the computational accuracy.

Since N. Aronszajn put forward the reproducing kernel space theory in 1950s, many researchers have done much works in this field. Especially in recent 20 years, more and more experts have seen the advantages of reproducing kernel space. And that the reproducing kernel space \(W_2^n[a,b]\) inherit all the properties of the Hilbert Space and possess some special and better properties, its widely application and better vision could be expected.

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