Balanced Multiwavelets and Application in Image Compression

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Abstract—Multiwavelet is attracting more and more attention for its good properties such as orthogonality, symmetry, compactly supported, high vanishing moment and balance contrast with the traditional wavelet. These properties, along with optimum time-frequency resolution (OPTFR) are all very important in image processing. Multiwavelet banks having all properties of symmetry, balancing, optimum time-frequency resolution, arbitrary order vanishing moment, orthogonality and compactly supported together are constructed for the first time. Application experiments in image compression showed that, most of our multiwavelet banks outperform the best tools known as multiwavelet SA4 or approximate wavelet DB4 in compression with texture image.

Index Terms—multiwavelet bank, symmetry, balanced, vanishing moment, time-frequency resolution, regularity

I. INTRODUCTION

Multiwavelet is attracting more and more attention for its good properties such as orthogonality, symmetry, compactly supported, high vanishing moment and balance contrast with the traditional wavelet. These properties, along with optimum time-frequency resolution (OPTFR) are all very important in image processing. In 1998, Jiang[4] first introduced the concept of time-frequency resolution of multiwavelets. Using OPTFR orthogonal multiwavelets, Jiang got some ideal result in image compressing. To present, Jiang’s work is known as the best in dealing with OPTFR multiwavelet and its application. However, with only 2-order vanishing moment, the capability of approximating to smooth function for Jiang’s OPTFR orthogonal multiwavelets is limited. Moreover, the orthogonal OPTFR multiwavelets in Jiang[4] are not balanced. Although we can get balanced orthogonal scaling functions and multiwavelets by rotating them for π/4 angle, the symmetry of the scaling functions, multiwavelets and their multifilter banks would be lost, which are not optimum in its time-frequency property any longer.

With the properties of symmetry, balancing, optimum time-frequency resolution, arbitrary order vanishing moment, orthogonality and compactly supported synthetically concerned, construction method of multiwavelets is improved, and balanced symmetric orthogonal multiwavelet banks with arbitrary vanishing moment and optimum in time-frequency resolution were obtained. Furthermore, application in image denoising and compression showed their advantages.

II. RELEVANT THEORY

Assume that $P$, $Q$ are $r \times r$ matrix filter with matrix coefficient $P_0$ and $Q_0$ satisfying $P_0 = 0$, and $Q_0 = 0$ $(k < 0$ and $k > N, N \in \mathbb{Z})$, and $\Phi = (\phi_1, \ldots, \phi_m)^T$, $\Psi = (\psi_1, \ldots, \psi_m)^T$ are compactly supported refinable vector-valued function satisfying

$$\Phi(x) = 2 \sum_{j=k}^{N} P j \Phi(2x-j), \Psi(x) = 2 \sum_{j=k}^{N} Q j \Phi(2x-j)$$

(1)

or equivalently satisfying

$$\tilde{\Phi}(\omega) = P(\omega/2)\tilde{\Phi}(\omega/2), \tilde{\Psi}(\omega) = P(\omega/2)\tilde{\Psi}(\omega/2),$$

(2)

where $P(\omega) = \sum_{j=k}^{N} P j e^{\omega j}$, $Q(\omega) = \sum_{j=k}^{N} Q j e^{\omega j}$.

Let $H_s$ denote the space of all $r \times r$ matrices with trigonometric polynomial entries whose Fourier coefficients are real and supported in $[1-N,N-1]$. The transition operator $T_s$ corresponding to $P$ is defined on $H_s$ by

$$T_s H(\omega) := P(\omega) H(\omega) + P(\omega + \pi) H(\omega + \pi), H \in H_s$$

Thus, the representation matrix $T_s$ of the operator $T_s$ is

$$T_s = (2A_{i,j})_{i,j=0,1,2,3},$$

where $A_s := \sum_{i=0}^{N} P_{i,j} \Theta P$.

Proposition 1: It is said that $P$ satisfies the vanishing moment conditions of order $m$ if there exist real $1 \times r$ row vectors with $\ell_r^* \neq 0$, $0 \leq \beta < m$, such that

$$\sum_{\alpha=0}^{\beta} \left( \begin{array}{c} \beta \\ \alpha \end{array} \right) (2i)^{-\beta} \ell_r^* D^{\alpha+\beta} P(0) = 2^{\alpha} \ell_r^*,$$

$$\sum_{\alpha=0}^{\beta} \left( \begin{array}{c} \beta \\ \alpha \end{array} \right) (2i)^{-\beta} \ell_r^* D^{\alpha+\beta} P(\pi) = 0,$$

where $D^{\alpha+\beta} P(\omega)$ denotes the matrix formed by the $(\beta - \alpha)$th derivatives of the entries of $P(\omega)$.

Symmetric properties of multiwavelet banks are very important in image applications. In this paper, we will discuss orthogonal multifilter banks $\{ P, Q \}$ with $\gamma P$, $\gamma Q$, $\gamma \in \mathbb{Z}$. 

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can be obtained by using the window function \( g \) and the time-duration \( E \). Then, compute the corresponding window area out.

**Definition 1.** For a window function \( f \) (with some smoothness and decay at infinity), the center in the time domain \( T \) and the time-duration \( \Delta_f \) of \( f \) are defined by

\[
T = \int_{-\infty}^{\infty} f(t) \, dt, \quad \Delta_f = \int_{-\infty}^{\infty} |f(t)| \, dt
\]

where \( E = \int_{-\infty}^{\infty} |f(t)| \, dt \). If the Fourier transform \( \hat{f} \) of \( f \) satisfies \( \omega \hat{f}(\omega) \in L^1(\mathbb{R}) \), then the center in the frequency domain \( \bar{\omega} \) and the frequency-bandwidth \( \Delta_\omega \) of \( f \) are defined by

\[
\bar{\omega} = \int_{-\infty}^{\infty} \hat{f}(\omega) \, d\alpha / E, \quad \Delta_\omega = \int_{-\infty}^{\infty} |\hat{f}(\omega)| \, d\alpha / E
\]

where \( E = \int_{-\infty}^{\infty} |\hat{f}(\omega)| \, d\alpha \).

**Definition 2.** The sum of resolution cells \( \Delta_\omega, \Delta_f \) and \( \Delta_\omega, \Delta_f \) is called the areas of time-frequency window (also called the time-frequency window area)

\[
S = \sum \Delta_\omega, \Delta_f + \Delta_\omega, \Delta_f
\]

To construct multiwavelets with good time-frequency properties means to find scaling functions and multiwavelets that are less in window area \( S \).

**III. MAIN ALGORITHM AND RESULT**

A. New algorithm of designing balanced symmetric orthogonal multifilter banks

This paper aims to construct the multiwavelets with good properties of arbitrary order vanishing moment, time-frequency property and regularity through the method of genetic algorithm, parameterization and solving the nonlinear equations of vanishing moment conditions.

Step 1: Construct the parameterized representation of multifilter banks \( P \) and \( Q \) which satisfy the conditions of orthogonality and balanced of order 1. \( P \) and \( Q \) can be expressed as (3)\[6\].

Step 2: Find the parameter that has maximum Sobolev regularity by the method of genetic algorithm. Then, compute the corresponding window area out.

Step 3: Find the parameter that makes the minimum window area by the method of genetic algorithm. Then, compute the corresponding window area out.

Step 4: Derive the conditions of vanishing moment of each order out according to Proposition 1.

Step 5: Solve the nonlinear equations of vanishing moment conditions. Thus determine the parameter value of \( P \) and \( Q \). Then, compute the corresponding window area out.

Step 6: If the solution of vanishing moment condition is not unique, then

Select the parameter that makes the maximum Sobolev regularity.

Select the parameter that makes the minimum window area.

Select the parameter which makes good perform both in regularity and time-frequency properties.

Step 7: Verify that whether the representation matrix \( T_\omega \) of lowpass filter corresponding to the selecting parameters can satisfy the condition \( E \) \[9\].

After deep investigating into the relationship between the properties of energy moments in the time and frequency domain of scaling function, the time-frequency window area of the scaling function and multiwavelet which satisfies the vanishing moment conditions of order \( m (m \geq 2) \) can be obtained by using the algorithm of calculating the window area\[11\].

During the 4th step of the algorithm, the vanishing moment condition of each order with parameters are obtained. Thus the problem turns to the solving of nonlinear equations. In this paper, the classical Newton algorithm and secant algorithm are used, during which the error \( |e| < 10^{-14} \).

Although these multifilter banks only produce scaling functions and multiwavelet functions with approximate symmetry, the parameterized form of \( P, Q \) can not only satisfy the condition of 1-order balance, but also has \( \gamma \) free parameters, which is suitable to use in constructing the balanced symmetric orthogonal multifilter banks with vanishing moment of high order and smoothness.

B. Construction result

Table 1 shows the properties of the balanced symmetric orthogonal multifilter banks constructed in this paper when \( \gamma = 2, 3, 4 \). When the supported in \([0, 2\times\gamma+1]\) and the vanishing moment is N-order, \( O_\gamma VN \) and \( S_\gamma VN \) represent those multifilter banks of optimum time-frequency resolution and optimum regularity, and \( OS_\gamma VN \) represent those of good time-frequency resolution and regularity properties concerned together. Here, \( O_\gamma VN \) and \( OS_\gamma VN \) are provided by this paper for the first time, and \( S_\gamma VN \) are the same in properties to Jiang's \[6\].

Contrast with the OPTFTR multiwavelets of Jiang's \[4\], our multifilter banks in this paper have the advantages as follows

a) The multifilter banks with optimum time-frequency resolution constructed in this paper have the vanishing
moment of 3-order, comparing with the 2-order of Jiang’s.

**TABLE I. CONTRASTING OF PROPERTIES BETWEEN VARIOUS MULTIWAVELETS**

<table>
<thead>
<tr>
<th>Multiwavelet</th>
<th>Support</th>
<th>Regularity</th>
<th>Vanishing moment</th>
<th>Window area</th>
</tr>
</thead>
<tbody>
<tr>
<td>GHM</td>
<td>3</td>
<td>1.4999</td>
<td>2</td>
<td>8.1492</td>
</tr>
<tr>
<td>J_EX2</td>
<td>3</td>
<td>1.7470</td>
<td>2</td>
<td>9.2793</td>
</tr>
<tr>
<td>J_EX4</td>
<td>5</td>
<td>1.7018</td>
<td>2</td>
<td>8.9565</td>
</tr>
<tr>
<td>O2_V2</td>
<td>5</td>
<td>1.0000</td>
<td>2</td>
<td>5.7427</td>
</tr>
<tr>
<td>S2_V2</td>
<td>5</td>
<td>1.5379</td>
<td>2</td>
<td>5.7706</td>
</tr>
<tr>
<td>O3_V2</td>
<td>7</td>
<td>1.5842</td>
<td>2</td>
<td>7.1738</td>
</tr>
<tr>
<td>S3_V2</td>
<td>7</td>
<td>1.9984</td>
<td>2</td>
<td>7.6951</td>
</tr>
<tr>
<td>OS3_V2</td>
<td>7</td>
<td>1.9911</td>
<td>2</td>
<td>7.5803</td>
</tr>
<tr>
<td>O3_V3</td>
<td>7</td>
<td>1.9983</td>
<td>3</td>
<td>7.6961</td>
</tr>
<tr>
<td>S3_V3</td>
<td>7</td>
<td>2.0953</td>
<td>3</td>
<td>9.3950</td>
</tr>
<tr>
<td>OS3_V3</td>
<td>7</td>
<td>1.9883</td>
<td>3</td>
<td>7.7421</td>
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<tr>
<td>O4_V2</td>
<td>9</td>
<td>1.0000</td>
<td>2</td>
<td>5.6579</td>
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<td>9</td>
<td>1.9967</td>
<td>2</td>
<td>6.4761</td>
</tr>
<tr>
<td>OS4_V2</td>
<td>9</td>
<td>1.9082</td>
<td>2</td>
<td>5.7589</td>
</tr>
<tr>
<td>O4_V3</td>
<td>9</td>
<td>1.5890</td>
<td>3</td>
<td>5.7281</td>
</tr>
<tr>
<td>S4_V3</td>
<td>9</td>
<td>2.3583</td>
<td>3</td>
<td>12.1789</td>
</tr>
<tr>
<td>OS4_V3</td>
<td>9</td>
<td>2.0460</td>
<td>3</td>
<td>7.3696</td>
</tr>
</tbody>
</table>

b) J_EX2 and J_EX4 are the multiwavelets listed in Jiang’s [4] as example 2 and 4. These orthogonal OPTFR multiwavelets are not balanced. Although we can further make them balanced by reversing the original multiwavelets for \( \pi/4 \) angle, the symmetry of the balanced multiwavelets and the multifilter banks would be lost, which are not optimum in its time-frequency property any longer. The multifilter banks constructed in this paper are not only balanced, but also have the balanced optimum time-frequency resolution and the symmetry properties as concerned in formula (3).

c) The window area of multifilter banks we constructed is smaller than those of Jiang’s [4]. For example, the window area of O2_V2 is 3.2 smaller than J_EX4, with their support being both 5. Although the support of O3_V2, O3_V3, OS3_V3, O4_V2, OS4_V2, O4_V3 and OS4_V3 are longer than Jiang’s, the window areas of them are all smaller by contraries.

IV. APPLICATION IN IMAGE COMPRESSION

In order to test the balanced symmetric orthogonal multiwavelets constructed in our research, we use the standard 512 \( \times \) 512 grey image named Lena, Baboon and Barbara to do the simulating experiment, in which Barbara and Baboon are texture images, and Lena is the typical smooth image. As i) wavelets CDF9-7, DB4 and multiwavelet SA4 performing well in image processing, ii)multiwavelet GHM being similar in form of multfilter bank with ours, iii) J_EX2 and J_EX4 being the best two OPTFR multiwavelets of Jiang’s[4], we choose these wavelets/multiwavelets to do our experiment.

During the experiment, the multiwavelet is decomposed to 5 degree first. Then, job of quantification and compression are done by the improved SPIHT algorithm [10]. Finally, decoding for SPIHT bit current and image reconstructing through multiwavelet inverse transformation are performed. As the aim of our experiment is to contrast with the different performances of various multiwavelets/wavelets, the process of arithmetic coding for bit current which is coded by SPIHT is neglected. The contrasting experiment based on SPIHT bit current can well demonstrate the difference performances of the multifilter/filter banks. Table II shows the values of PSNR for the multiwavelets/wavelets dealing with Barbara, Baboon and Lena with the compression rate of 16:1, 32:1 and 64:1. As shown in fig. 1, the summary of compression appreciation index (adding three PSNRs in the same row of Table II) to every multiwavelet for Barbara, Baboon and Lena are listed.

From the experiment we found that:

a) The performances of the multifilter banks we constructed in image compression are obviously better than that of GHM.

b) From Table II we know, as for the three images we chose, most of our multifilter banks performed better than J_EX2 in image compression. As J_EX2 being the OPTFR multiwavelet of Jiang’s[4] with shortest support in [0,3], most of our multifilter banks performed better than J_EX2 in compression with Barbara, and approximate J_EX2 with Baboon.

c) As shown in Table II, the best performances in compressing with texture image Barbara and Baboon are made by wavelet CDF9-7, wavelet DB4 and multiwavelet SA4. However, most of our multiwavelets are synthetically better performed than the best multiwavelet SA4, and better than or approximate that of wavelet DB4.

Image compression experiments show that, the performances of most of our multiwavelets are better than that of SA4 and approximate wavelet DB4.

It is not difficult to see that, the performances of OS3_V2, S3_V2, O3_V3, S3_V3, O3_V3, OS3_V3 in image compression are better than that of the others especially when dealing with texture images.

V. CONCLUSION

In this paper, construction method is improved to obtain the multifilter banks that have good properties and better performances in image compression. By introduce the time-frequency property to the construction of balanced symmetric orthogonal multifilter banks, we first obtained the optimum time-frequency resolution multifilter banks which have arbitrary vanishing moment. With the properties of vanishing moment, regularity, time-frequency resolution, compactly supported synthetically concerned, we first constructed the multifilter banks which have good performances both in regularity and time-frequency resolution. Furthermore, the smoothest balanced symmetric orthogonal multifilter
bank is obtained. Application experiments in image compression show that, their performances are better than that of SA4 and approximate wavelet DB4.

TABLE II. PSNR OF WAVELETS/MULTI WAVELETS IN COMpressING DIFFERENT COMPRESSING RATIO

<table>
<thead>
<tr>
<th>NO.</th>
<th>Wavelet/ multirwavelet</th>
<th>Barbara</th>
<th>Baboon</th>
<th>Lena</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>GHM</td>
<td>28.1833</td>
<td>25.1393</td>
<td>23.3500</td>
</tr>
<tr>
<td>2</td>
<td>J EK2</td>
<td>29.7339</td>
<td>26.1370</td>
<td>23.6241</td>
</tr>
<tr>
<td>3</td>
<td>J EK4</td>
<td>26.3333</td>
<td>24.0995</td>
<td>22.2634</td>
</tr>
<tr>
<td>4</td>
<td>SA4</td>
<td>29.5402</td>
<td>26.0751</td>
<td>23.6423</td>
</tr>
<tr>
<td>5</td>
<td>D84</td>
<td>29.9789</td>
<td>26.2356</td>
<td>23.6681</td>
</tr>
<tr>
<td>6</td>
<td>CDF9-7</td>
<td>30.3255</td>
<td>26.4018</td>
<td>23.7980</td>
</tr>
<tr>
<td>7</td>
<td>O2 V2</td>
<td>29.7733</td>
<td>26.1491</td>
<td>23.5683</td>
</tr>
<tr>
<td>8</td>
<td>S2 V2</td>
<td>29.7684</td>
<td>26.1465</td>
<td>23.5672</td>
</tr>
<tr>
<td>9</td>
<td>O3 V2</td>
<td>29.3660</td>
<td>26.1106</td>
<td>23.5552</td>
</tr>
<tr>
<td>10</td>
<td>S3 V2</td>
<td>29.8516</td>
<td>26.4343</td>
<td>23.7064</td>
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<tr>
<td>11</td>
<td>OS3 V2</td>
<td>29.7959</td>
<td>26.3964</td>
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<td>26.4346</td>
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<td><strong>23.7693</strong></td>
</tr>
<tr>
<td>14</td>
<td>OS3 V3</td>
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<td>26.4444</td>
<td>23.7158</td>
</tr>
<tr>
<td>15</td>
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<td>26.2353</td>
<td>23.6407</td>
</tr>
<tr>
<td>16</td>
<td>S4 V2</td>
<td>29.7660</td>
<td>26.2629</td>
<td>23.6102</td>
</tr>
<tr>
<td>17</td>
<td>OS4 V2</td>
<td>30.0384</td>
<td>26.3106</td>
<td>23.6948</td>
</tr>
<tr>
<td>18</td>
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<td>26.1538</td>
<td>23.5989</td>
</tr>
<tr>
<td>19</td>
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<td>29.8048</td>
<td>26.2381</td>
<td>23.6644</td>
</tr>
<tr>
<td>20</td>
<td>OS4 V3</td>
<td>29.8183</td>
<td>26.4303</td>
<td>23.7256</td>
</tr>
</tbody>
</table>

TABLE II. PSNR OF WAVELETS/MULTI WAVELETS IN COMpressING DIFFERENT COMPRESSING RATIO

Figure 1. Compressing performance of different Wavelet/Multiwavelet

REFERENCES


