A New T-F Function Approach for Discrete Global Optimization

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Abstract—**The T-F function method is an approach to find the global minimum of a multidimensional function. This paper gives a new definition of T-F function for discrete global optimization. A T-F function satisfying this definition is proposed. Furthermore, we discuss the properties of the proposed T-F function and design a new discrete T-F function algorithm. Numerical results on several test problems indicate that the proposed algorithm is reliable and efficient.**

Index Terms—**Discrete global optimization, Filled function, Tunnel function, T-F function.**

I. INTRODUCTION

Consider the following discrete global minimization problem: (P) $\min_{x\in X}$ f(x),

where $f: X \subset Z^n \to R^1$, and Z^n is the set of integer points in R^n, This problem is important since we can find a variety of practical problems in which discrete global minimizer has to be obtained. The difficulties for searching a global optimum present at two sides: how to leave from a local minimizer to a smaller one and how to judge the current minimizer is a global one.

During the past three or four decades, many theories and algorithms in discrete global optimization have been developed (see [2,3,4,5]). Among these methods, a practical methods for discrete global optimization is discrete filled function method introduced by [1] and farther developed by [6,7,8,9,10] to overcome the first difficulty.

In this paper, we propose a new T-F function method to solve a discrete global minimization problem over a box domain. The method iterates from one local minimum to a better one. In each iteration, we construct a T-F function that attains strict local maximum at the current solution. A local minimizer of the T-F function leads to a new solution of reduced value. Iteration follows in this manner to reach a global minimizer. Some promising computational are reported.

The paper is organized as follows: After tis introduction, we present some basic knowledge of discrete global optimization in Section 2 and introduce a new T-F function and study its properties in Section 3. Then we propose a T-F function solution method in Section 4 and report the computational results on the proposed method in Section 5. At last, we give the conclusion.

II. PRELIMINARIES

In this section, we first recall some definitions for nonlinear integer programming and then define a discrete tunnel function and T-F function.

Definition 1: A point $x^* \in X \subset Z^n$ is called a discrete local minimizer of $f(x)$ over X , if $f(x) \ge f(x^*)$ for any $x \in X \bigcap N(x^*)$; If $f(x) \ge f(x^*)$ for all $x \in X$, then x^* is called a discrete global minimizer of $f(x)$; If $f(x) > f(x^*)$ holds for all $x \ne x^*$, $x \in X \bigcap N(x^*)$, then x^* is called a strict discrete local minimizer of $f(x)$. Where $N(x) = \{x, x \pm e_i, i = 1, 2, ...n\}$, e_i is the ith unit vector (the n dimensional vector with the ith component equal to one and all other components equal to zero).

Denotes

$D = \{\pm e_i : i = 1, 2, ...n\}$, $D_x = \{d \in D : x + d \in D\}$.

Algorithm 2.1(Discrete local minimization method)

1. Start from an initial point $x \in X$.

2. If x is a local minimizer of $f(x)$, then stop; Otherwise, let

$$d^* = \arg\min_{d_i \in D_i} \{f(x + d_i) : f(x + d_i) < f(x)\}.$$

3. Let $x = x^* + d$, go to Step 2.

Definition 2: $P(x,x^*)$ is called a discrete filled function of $f(x)$ at a discrete local minimizer x^* if it has the following properties:

1. x^* is a strict discrete local maximizer of $P(x,x^*)$.
2. $f(x)$ has no discrete local minimizer in region $S_1=\{x\in X\setminus x^*: f(x)\geq f(x^*)\}$.
3. If x^* is not a discrete global minimizer of $f(x)$, then $P(x,x^*)$ does have a discrete minimizer in region $S_2=\{x\in X: f(x)<f(x^*)\}$.

Definition 3: $P(x,x^*)$ is called a discrete tunnel function of $f(x)$ at x^* if, for any $x^0\in X$ with $r>0$, $P(x,x^*)=0$ if and only if $f(x^0)-f(x^*)+r\leq 0$.

Definition 4: $P(x,x^*)$ is called a T-F function of $f(x)$ at x^* if it is both a discrete tunnel function and a discrete filled function.

III. A NEW DISCRETE T-F FUNCTION

To continuce, we need the following assumptions :

Assumption 1: For any $x\in Z^n\setminus X, f(x)=+\infty.$

This implies: $\min_{x\in X} f(x)=\min_{x\in Z^n} f(x).$

Assumption 2: $f:\bigcup_{x\in X}N(x)\to R^1$ satisfies the Lipschitz condition, that is, there exists a constant $L>0$ such that for any $x,y\in\bigcup_{x\in X}N(x)$, it holds $|f(x)-f(y)|\leq L\|x-y\|$.

In order to present a new T-F function, for any $r>0$, we first introduce a function $h_r(t)$ as follows:

$$h_r(t)=\begin{cases}1 & t\geq 1+r\\ \phi_r(t) & 1\leq t<1+r\\ 0 & t<1\end{cases}$$

Where $\phi_r(t)$ satisfies the following condition: $\phi_r(1)=0,\phi_r(r+1)=1$ and $0\leq\phi_r(t)\leq 1$ for $1<t<1+r$.

Now we give a auxiliary function $T(x,x^*,r,q)$ of $f(x)$ at the current local minimizer x^* with $r>0$ and $q>0$ as follows:

$$T(x,x^*,r,q)=\frac{1}{1+\|x-x^*\|}h_r(f(x)-f(x^*)+r+1)$$
$$+q\max(f(x)-f(x^*)),$$

where $0<r<\min_{f(x_1)\neq f(x_2),x_1,x_2\in X}|f(x_1)-f(x_2)|$.

Obviously $T(x,x^*,r,q)$ is tunnel function, the following theorems show that it is also a filled function, so this function is a T-F function.

In the following, we assume that x^* is a local minimizer of $f(x)$, $K=\max_{x_1,x_2\in X}\|x_1-x_2\|$..

Theorem 1: If $0<q<(2L)^{-1}$, then x^* is a strict local maximizer of $T(x,x^*,r,q)$.

Proof: By the condition, we have

$$f(x^*+d)\geq f(x^*),$$
$$f(x+d)-f(x^*)+r+1\geq r+1$$

for any $x\in X\bigcap N(x^*)$; where. $d\in D$.

Therefore, we have

$$T(x^*+d,x^*,r,q)=\frac{1}{1+\|x^*-x^*+d\|}q[f(x^*+d)-f(x^*))]$$
$$\leq 0.5+qL<1=T(x^*,x^*,r,q),$$

which implies that x^* is a strict local maximizer of $T(x,x^*,r,q)$.

Lemma: For every $x'\in X$, there exists $d\in D$. such that $\|x'-x^*+d\|>\|x'-x^*\|$.

Proof: If there exists $i\in\{1,2,...,n\}$ such that $[x']_i\geq[x^*]_i$, where $[x]_i$ is the ith component of any vector x, then $d=e_i$; On the other hand, if there exists i such that $[x']_i\leq[x^*]_i$, then $d=-e_i$.

Theorem 2: Suppose that

$$q<[L(K+1)(K+2)(2K+1)]^{-1}, x\in S_{1,}$$

then x is not a local minimizer of $T(x,x^*,r,q)$.

Proof: For any $x\in S_{1,}$, by Lemma , there exists $d\in D$ with $x+d\in\bigcup_{x\in X}N(x)$ such that $\|x-x^*+d\|>\|x-x^*\|$ and $x+d\in X$. For the above d, consider the following two cases:

Case i: $f(x+d)\geq f(x)\geq f(x^*)$. In this case, since

$$\|x-x^*\|\leq K,\|x-x^*+d\|\leq K+1,$$

$\|x-x^*\|\|x-x^*+d\|^2-\|x-x^*\|^2\geq 1$, we have

$$\frac{-\|x-x^*\|+\|x-x^*+d\|}{(\|x-x^*\|+1)(\|x-x^*+d\|+1)}\geq$$
$$\frac{-\|x-x^*\|^2+\|x-x^*+d\|^2}{(\|x-x^*\|+1)(\|x-x^*+d\|+1)(\|x-x^*\|+\|x-x^*+d\|)}$$
$$\geq[(K+1)(K+2)(2K+1)]^{-1}.$$

Therefore,

$$T(x+d,x^*,r,q)-T(x,x^*,r,q)=\frac{-\|x-x^*\|+\|x-x^*+d\|}{(\|x-x^*\|+1)(\|x-x^*+d\|+1)}+q[f(x+d)-f(x)]$$
$$\le [(K+1)(K+2)(2K+1)]^{-1}+qL<0,$$

that is $T(x+d,x^*,r,q)<T(x,x^*,r,q)$.

Case ii: $f(x)>f(x+d)\ge f(x^*)$. In this case,

$$T(x+d,x^*,r,q)-T(x,x^*,r,q)=\frac{1}{1+\|x-x^*+d\|}-\frac{1}{1+\|x-x^*\|}+q(f(x+d)-f(x^*))<0.$$

Case iii:

$f(x+d)\ge f(x^*)>f(x+d)>f(x^*)-r$. In this case, we have

$$T(x+d,x^*,r,q)=\frac{1}{1+\|x-x^*+d\|}h_r(f(x+d)-f(x^*)+r+1)\le\frac{1}{1+\|x-x^*+d\|}<\frac{1}{1+\|x-x^*\|}+q[f(x)-f(x^*))]=T(x,x^*,r,q).$$

Case iv: $f(x)\ge f(x^*)>f(x+d)$ and $f(x+d)-f(x^*)+r\le 0$. In this case, we have

$$T(x+d,x^*,r,q)=0<\frac{1}{1+\|x-x^*\|}\le T(x,x^*,r,q).$$

The above four cases imply that if $q>0$ is small enough, then, for .any $x\in S_{1,}$ it is not a discrete local minimizer of $T(x,x^*,r,q)$.

Theorem 3: Suppose that x^* is not a discrete global minimizer of $f(x)$, then there exists a minimizer x_1^* of $T(x,x^*,r,q)$ in the region.

$$S_2=\{x\in X: f(x)<f(x^*)\}.$$

Proof: Let $x^{'}$ be a global minimizer of (P). By the condition, we have $f(x^{'})-f(x^*)+r+1<1$. Thus $T(x^{'},x^*,r,q)=0$. On the other hand , for any x , we have $T(x,x^*,r,q)\ge 0$ by the definition of $h_r(t)$. Therefore $x^{'}$ is a minimizer of $T(x,x^*,r,q)$.

IV. SOLUTION ALGORITHM

Based on the foregoing properties of the T-F function in the previous section, we propose the following discrete T-F function algorithm.

Algorithm 2 (Discrete T-F function method)

1. Input the lower bound of r , namely $r_L=1e-8$; . Input an initial point $x_0^{(0)}\in X$; Let $D=\{\pm e_i: i=1,2,...n\}$.

2. Starting from an initial point $x_0^{(0)}\in X$, minimize $f(x)$ and obtain the first local minimizer x_0^*; . Set $k=0, q=1, r=1$.

3.Set $x_k^{(0)i}=x_k^*+d_i,\quad d_i\in D,\quad i=1,2,...,2n,$ $J=[1,2,...,2n], j=1.$

4. Set $i=J_j, x=x_k^{(0)i}$.

5. If $f(x)<f(x^*)$, then use x as initial point for local minimization method to find x_{k+1}^* such that $f(x_{k+1}^*)<f(x_k^*)$. Set $k=k+1$, go to step3.

6. Let $D_0=\{d\in D: x+d\in X\}$. If there exists $d\in D$ such that $f(x+d)<f(x_k^*)$, then use $x+d^*$, where $d^*=\arg\min_{d\in D_0}\{f(x+d)\}$, as an initial point for a local minimization method to find x_{k+1}^* such that $f(x_{k+1}^*)<f(x_k^*)$. Set $k=k+1$, go to step3

7. Let $D_1=\{d\in D_0: \|x+d-x_k^*\|>\|x-x_k^*\|\}$.
If $D_1=\varnothing$, goto 10; If there exists $d\in D_1$ such that $T(x+d,x_k^*,r,q)\ge T(x,x_k^*,r,q)$, let $q=0.1q,\ J=[J_j,....,J_{2n},J_1,...J_{j-1}],\ j=1$, goto 4.

8. Let
$D_2=\{d\in D_1: f(x+d)<f(x), T(x+d,x_k^*,r,q)<T(x,x_k^*,r,q)\}$. If $D_2\ne\varnothing$, let
$d^*=\arg\min_{d\in D_2}\{f(x+d)+T(x+d,x_k^*,r,q)\}$;
Otherwise, let $d^*=\arg\min_{d\in D_1}T(x+d,x_k^*,r,q)$; let $x=x+d$, goto 6.

9. If $i<2n$, Let $i=i+1$, got 4.

10. Let $r=0.1r$. If $r>r_L$, goto 3; Otherwise, algorithm stops and x_k^* is a discrete global minimizer.

V. NUMERICAL EXPERIMENT

In the following, computational results of several test problems using the above algorithm are summarized. The experiment is programmed by Fortan95. The symbols are used in the tables are listed as follows:

x_k^* :The k-th initial point.

k :The iteration number.

$f(x_k^*)$: The function value of k-th minimizer.

TI : :The CPU time in seconds for algorithm to stop.

PN : The N-th problem.

DN : The dimension of objective function.

IN : The number of iteration cycles.

FN : The number of computing $f(x)$ to stop.

Example 1

$$\text{Min } f(x) = (x_2 - x_1^2)^2 + (1-x_1)^2 + 90(x_4 - x_3^2)^2 + (1-x_3)^2 + 10.1[(x_2-1)^2 + (x_4-1)^2] + 19.8 (x_2-1)(x_4-1),\ |x_i| \le 10, x_i \text{ is integer}, i = 1,2,3,4.$$

This problem has 21^4 feasible points and many local minimizers, but only one global minimum solution: $x^*_{global} = (1,1,1,1)$ with $f(x^*_{global}) = 0$. We used 5 initial points in our numerical test: (9,6,5,6),(10,10,10,10),-(10,10,10,10),(-10,10,-10,10),(10,10,-10,10).For every test, the proposed algorithm can identify the global minimum. Numerical results are listed in table 1 and 3.

Example 2

$$\text{Min } f(x) = (1 + (x_1 + x_2 + 1)^2 (19 - 14x_1 + 3x_1^2 - 14x_1 + 6x_1x_2 + 3x_2))(30 + (2x_1 - 3x_2)^2 (18 - 32x_1 + 12x_1^2 + 48x_2 - 36x_1x_2 + 27x_2^2)), x_i = 0.001y_i,\ |y_i| < 2000, y_i \text{ is integer}, i = 1,2.$$

This problem has 4001^2 feasible points and many local minimizers, but only one global minimum solution: $x^*_{global} = (0,-1)$ with $f(x^*_{global}) = 3$. We used 5 initial points in our numerical test: (2000,2000),-(2000,2000),(1196,1156),(-2000,2000),(2000,-2000). For every test, the proposed algorithm can identify the global minimum. Numerical results are listed in table 2 and 3.

Example 3

$$\text{Min } f(x) = (x_1 - 1)^2 + 5(x_n - 1)^2 + \sum_{i=1}^{n} (n-i)(x_i^2 - x_{i+1})^2,\ |x_i| \le 5, x_i \text{ is integer}, i = 1,2,...,n.$$

This problem has 11^n feasible points and many local minimizers, but only one global minimum solution: $x^*_{global} = (1,...,1)$ with $f(x^*_{global}) = 0$. For n=25,50,100, we used initial points (5,…,5) in our numerical test. For every test, the proposed algorithm can identify the global minimum. Numerical results are listed in table 3.

Example 4

$$\text{Min } f(x) = \sum_{i=1}^{n-1} [100(x_{i+1} - x_i^2)^2 + (1-x_i)^2,\ |x_i| \le 5, x_i \text{ is integer}, i = 1,2,...,n.$$

This problem has 11^n feasible points and many local minimizers, but only one global minimum solution: $x^*_{global} = (1,...,1)$ with $f(x^*_{global}) = 0$. For n=25,50,100, we used initial points (5,…,5) in our numerical test. For every test, the proposed algorithm can identify the global minimum. Numerical results are listed in table 3.

TABLE I. *EXAMPLE 1*

k	x_k^0	$f(x_k^0)$	x_k^*	$f(x_k^*)$
1	(9,6,5,6)	596070.0	(2,4,2,3)	342.1000
2	(1,1,2,3)	134.4000	(1,1,2,4)	91.7000
3	(1,1,0,1)	91.0000	(1,1,1,1)	0.0000

TABLE II. EXAMPLE 2

k	y_k^0	$f(y_k^0)$	y_k^*	$f(y_k^*)$
1	(1196,1156)	1862.019	(1278,888)	954.1411
2	(1280,889)	954.1388	(1279,889)	654.1375
3	(388,-25)	953.7714	(-600,-400)	30.0000
4	(-271,-720)	29.98818	(0,-1000)	3.0000

TABLE III. SUMMARY

PN	DN	IN	TI	FN
1	4	3	0.1301984	85431
2	2	4	2.693856	2077634
3	25	2	79.68953	16350697
3	50	2	1136.9578	143652497
3	100	2	5836.1332	353612364
4	25	2	29.7654	5984136
4	50	2	459.9721	38395618
4	100	2	8124.6938	271123684

VI. CONCLUSIONS

This paper gives a definition of the T-F function for nonlinear integer programming, and present a T-F function which containing two parameters. A discrete T-F function algorithm based on the theoretical properties of the proposed T-F function is designed. Reported numerical results indicate that this algorithm is erricient and reliable.

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