The Analysis and Design of Nearly-Orthogonal Symmetric Wavelet Filter Banks

Yun Wu
Shenzhen Graduate School of Peking University, Shenzhen, P. R. China
Email: wuyun_v@126.com

Yong Zhao and Ling Shen
Shenzhen Graduate School of Peking University, Shenzhen, P. R. China
Email: zhaoyong@szpku.edu.cn, forrest.lingshen@gmail.com

Abstract—**The design and analysis of nearly-orthogonal symmetric wavelet filter banks has been studied. Methods for analyzing the correlation of the nearly-orthogonal filter banks are proposed. The basic idea is to impose multiple zeros at the aliasing frequency to a symmetric filter, minimize the deviation of the filter satisfying the orthogonal condition, and a nearly orthogonal filter bank can be obtained. Since multiple zeros are imposed, a scaling function may be generated from the minimized filter. The integer translates of the wavelet and the scaling functions are nearly orthogonal. The integer translates of the wavelet at different scale are completely orthogonal. We will construct a perfect reconstructed semi-orthogonal filter bank. Detailed analysis of correlation of the nearly-orthogonal filter banks are given in this paper.**

Index Terms—**wavelet, filter banks, nearly-orthogonal, symmetric**

I. INTRODUCTION

Sub-band transforms have been successfully used in many areas of signal processing, especially in the compression of still image and video signals. This technique normally employs analysis/synthesis multi-rate digital filter banks for the decomposition and reconstruction of signals [1],[2]. It has been shown that filter banks are closely connected with the wavelet transform which is a decomposition of a function (signal) into a set of basis functions consisting of contractions, expansions, and translations of a mother function [1]. Lots of wavelet transform decompositions of functions can be implemented as filter banks, and on the other hand, filter banks can be used to generate wavelet base if they satisfy some basic condition such as the perfect reconstruction and regularity.

Using wavelet filter banks for image processing has been studied widely and it has been shown to have potential for image coding [3]-[7]. While most of these developments concentrated on one dimensional signals and the multidimensional case was handled via the tensor product, some of the more recent efforts concentrated on the "true" multidimensional case, both from the filter bank and the wavelet aspects [8]-[13]. Regularity is a crucial distinction between an ordinary filter bank and a wavelet filter bank and it has been shown in [1]-[6] that the vanishing moments and regularity are relevant to denoising [8], sub-band coding [14]-[16], etc., although it is not necessary to maximize the regularity of a filter bank in all applications [5],[6].

For two channel filter banks, no matter whether they are one-dimensional or two-dimensional, there is not one which can be simultaneously be symmetric (linear phase) [17], compactly supported (FIR) [18] and orthogonal, except the Harr wavelet [19] which is not continuous. However, linear phase, which is good properties for filtering to avoid distortion in signal processing, is often desired. To obtain a FIR filter bank, only nearly-orthogonal filter bank is possible. In this paper, zeros at aliasing frequency points are directly imposed on the two-dimensional quincunx symmetric filter, and optimize it to satisfy the orthogonal condition as closely as possible. Then a nearly-perfect reconstructed (NPR) [20] and nearly-orthogonal (ON) filter bank can be obtained, and a corresponding scaling function can be generated. The question is what is the corresponding wavelet and what is the degree of orthogonality of it, since the filter bank is not perfect reconstructed. Firstly, we can construct a semi-orthogonal 1D which is perfect reconstructed. We will show that the semi-orthogonal filter is very close to the nearly-orthogonal filter bank in the sense that the corresponding filters in the two filter bank have the exact phase response and almost the same amplitude response. The NPR filter bank is regarded as an implementation of the semi-orthogonal filter and correlation analysis is carried on the semi-orthogonal filter bank. Both theoretical and experimental analysis are given, which show that the semi-orthogonal filter bank is nearly-orthogonal. Secondly, a complementary filter with the same vanishing moment as the original filter is found. The correlation of this perfect reconstructed filter bank is also carried out. The analysis show that, in the

corresponding author: Yun Wu. Room 207 Building A, Peking Campus, University Town, Shenzhen, Guangdong, 518055, P.R.China

biorthogonal wavelet constructed process, orthogonality of the wavelet at different scale become worse obviously.

This paper is organized as follows. Two Channel Filter Banks and Wavelets are briefly reviewed in Section II. In Section III, Symmetric Nearly-Orthogonal Filter Banks are presented in detail. In Section IV, the Correlation Analysis of Wavelet Filter Banks is discussed completely, and some design examples are shown. Finally, conclusions are given in Section V.

II. TWO CHANNEL FILTER BANKS AND WAVELETS

It is known that there is a close relation between filter banks and wavelets. A lot of discrete wavelet transforms can be implemented as filter banks, and perfect reconstructed filter banks may generate wavelets, provided that they satisfy some regularity condition. The condition for a filter bank being capable to generate wavelet bases is very complicated, as indicated in [21]. However, a necessary condition for a filter can be used to generate a scaling function is that it must have some zeros [22],[23] at the point of $z=-1$. If this condition is satisfied, the filter bank may be capable to generate a multiresolution analysis and a wavelet basis [24].

A general one-dimensional two channel filter bank is shown in Fig. 1.

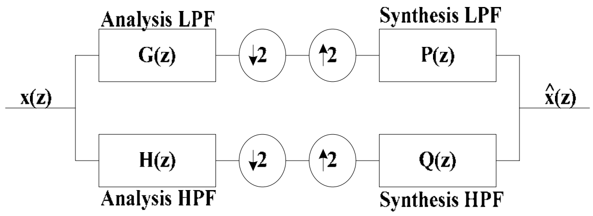

Figure 1. One dimensional two-channel filter bank

The output is

$$y(z)=\frac{1}{2}\left[P(z)G(z)+Q(z)H(z)\right]x(z)$$
$$+\frac{1}{2}\left[P(-z)G(z)+Q(-z)H(z)\right]x(-z) \qquad (1)$$

If it is perfect reconstructed (PR), we would have $P(z)G(z)+Q(z)H(z)=2$ and $P(-z)G(z)+Q(-z)H(z)=0$. If we switch the analysis filter bank and synthesis filter, it is still PR. If a filter bank is PR, the synthesis low-pass filter $P(z)$ have some zeros at the point of $z=-1$, and $P(1)=\sqrt{2}$, then it may generate a multiresolution analysis and a wavelet basis with a scaling function as

$$\hat{\phi}(\omega)=\prod_{k=1}^{\infty}\frac{1}{\sqrt{2}}P(e^{j\omega}) \qquad (2)$$

The notation "hat" above a variable is used to denote its Fourier Transform. And the wavelet can obtained from the two-scale equations

$$\varphi(t)=\sqrt{2}\sum_k p(k)\varphi(2t-k) \qquad (3)$$

$$\psi(t)=\sqrt{2}\sum_k q(k)\varphi(2t-k) \qquad (4)$$

where $p(k), q(k)$ are the coefficients of the polynomial of $P(z)$ and $Q(z)$ respectively, i.e.,

$$P(z)=\sum_k p(k)z^{-k} \qquad (5)$$

$$Q(z)=\sum_k q(k)z^{-k} \qquad (6)$$

In frequency domain, we have

$$\hat{\phi}(\omega)=\frac{1}{\sqrt{2}}P(e^{j\omega/2})\hat{\phi}(\omega/2) \qquad (7)$$

$$\hat{\psi}(\omega)=\frac{1}{\sqrt{2}}Q(e^{j\omega/2})\hat{\phi}(\omega/2) \qquad (8)$$

Similarly, the synthesis filter bank is given.

III. SYMMETRIC NEARLY-ORTHOGONAL FILTER BANKS

Consider filter bank with the synthesis low-pass filter (LPF), analysis LPF, synthesis high-pass filter (HPF) and analysis HPF [25] as

$$P_{no}(z)=P(z) \qquad (9)$$

$$G_{no}(z)=P(z^{-1}) \qquad (10)$$

$$Q_{no}(z)=-z^{-1}P(-z^{-1}) \qquad (11)$$

$$Q_{no}(z)=-zP(-z) \qquad (12)$$

it is easy to verify that response to the aliasing $x(-z)$ is completely cancelled because the output is equal to (13)

$$y(z)=\frac{1}{2}(P(z)P(z^{-1})+(-z^{-1})P(-z^{-1})(-z)P(-z))x(z)$$
$$+\frac{1}{2}(P(-z)P(z^{-1})+(z^{-1})P(z^{-1})(-z)P(-z))x(-z)$$
$$=\frac{1}{2}(P(z)P(z^{-1})+(-z^{-1})P(-z^{-1})(-z)P(-z))x(z)$$

If $P(z)$ satisfies

$$P(z)P\left(z^{-1}\right)+P(-z)P\left(-z^{-1}\right)=2 \qquad (14)$$

then the filter bank would be a perfect reconstructed and an orthogonal filter bank. If we don't assume that $P(z)$ is symmetric, we can obtain orthogonal filter as in [23]. However, it is known that there is not a two-channel filter which can be simultaneously symmetric, orthogonal, and FIR [1]. If we assume the filter $P(z)$ is symmetric, then (14) can only be approximately satisfied. Assume that a filter $P(z)$ has $2m$ zeros at $z=-1$, then it can be written as follows (15)

$$P(z)=(\frac{1+z^{-1}}{2})^m(\frac{1+z}{2})^m(c_M z^{-M}++\cdots+c_0+\cdots+c_M z^M)$$

where $c_M, c_{M-1}, \cdots, c_0$ are the parameters of the remainder polynomial and $2M+1$ is the length of the remainder polynomial. To design a nearly orthogonal filter banks, it should make $P(z)$ satisfy the (14) as accurately as possible. Usually, we can minimize the following cost function to get a nearly-orthogonal filter, i.e.

$$minimize_C \int_0^{\pi} |P(e^{j\omega}) + P(e^{j(\omega+\pi)}) - 2| d\omega \quad (16)$$

But this cost function involves much more computation, hence we use another criterion. Rewrite (14) as

$$(\mathbf{P} * \overline{\mathbf{P}} + W\mathbf{P} * \overline{\mathbf{W}}\,\mathbf{P} - 2\mathbf{I}_0)\mathbf{z} = 0 \quad (17)$$

where * is convolution operation,

$$\mathbf{z} = [z^{-2N}, \cdots, z^{-1}, 1, z, \cdots, z^{2N}], \quad (18)$$

$$C_0 = [\frac{1}{4} \;, \frac{1}{2} \;, \frac{1}{4}]^T \quad (19)$$

$$C_1 = [c_M, \cdots, c_1, c_0, c_1, \cdots, c_M]^T \quad (20)$$

$$\mathbf{P} = \underbrace{(C_0 * \cdots * C_0)}_{m} * C_1 \quad (21)$$

$$N = M + m \quad (22)$$

To make $P(z)$ satisfy (17), we minimize

$$L = \| \mathbf{P} * \overline{\mathbf{P}} + W\mathbf{P} * \overline{\mathbf{W}}\,\mathbf{P} - 2\mathbf{I}_0 \| \quad (23)$$

where $\|\cdot\|$ can be any kind of norm of vector. Here it is the largest singular value of a vector or matrix. Through minimizing L with respect to the coefficients of $c_M, \cdots, c_1, c_0$, we obtain a symmetric nearly perfect reconstructed filter bank with a synthesis LPF as $P_{no}(z)$, analysis LPF as $G_{no}(z) = P(z^{-1})$, synthesis HPF as $Q_{no}(z) = -z^{-1}P(z^{-1})$, analysis HPF filter as $H_n(z) = -zP(-z)$.

The design for $m = 1 \sim 3, n = 2 \sim 5$ were carried out, and amplitude distortion $\frac{1}{2}|P(e^{j\omega}) + P(e^{j(\omega+\pi)}) - 2|$ in decibel for these cases are given in Table 1. For example, the amplitude distortion for the case of m=2, M=5, is shown in Fig. 2.

TABLE I.

MAXIMUM DISTORTION OF AMPLITUDE OF 1D NEARLY-ORTHOGONAL FILTER BANK.

	m=1	m=2	m=3
n=2	73.01	50.93	22.40
n=3	119.80	58.60	57.71
n=4	171.27	66.35	63.75
n=5	178.87	109.31	71.50

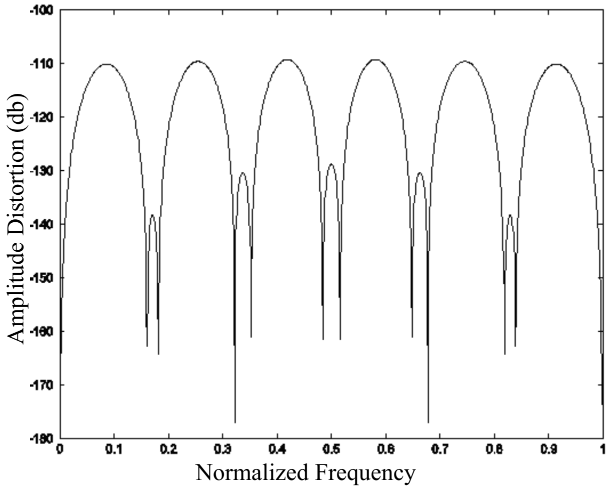

Figure 2. Amplitude distortion of 1D nearly-orthogonal filter bank of m=2, M=5

Since there are a number of zeros at the point of $z = -1$, $P(z)$ may generate a scaling function $\varphi(t) = \lim \varphi_n(t)$ using the following dilation equation [25],[26]

$$\varphi_n(t) = \sum_{k=0}^{n} p(k)\varphi_{n-1}(2t - k), \quad (24)$$

starting from any $\varphi_0(t)$ with $\hat{\phi}(0) = 1$. If the order of $P(z)$ is $\{-n_1, n_2\}$, the support of the scaling function will not exceed the interval $(-n1_1, n_2)$. The convergence can be investigated by observing the convergence of the above recursion. Using the above equation to calculate the value of $\varphi_n(t)$, the computation increase exponentially as n increase. However, we can always observe the convergence on the discrete points of $\frac{l}{L}$, where $\frac{l}{L}$ is the sample interval since the calculation of $\varphi_n(t)$ on these discrete points only involving values of $\varphi_{n-1}(t)$ on the same discrete points.

$$\varphi_n(l/L) = \sum_{k=0}^{n} p(k)\varphi_{n-1}(\frac{2l - L \cdot k}{L}) \quad (25)$$

This allows us the compute $\varphi_n(t)$ for large n to observe the convergence. For example, consider the case: m=1, M=5, $\varphi_n(t)$ when n= 2,5,10,100 are computed on the discrete points of $\{\frac{i}{32}, i \in Z\}$ and are shown in Fig. 3.

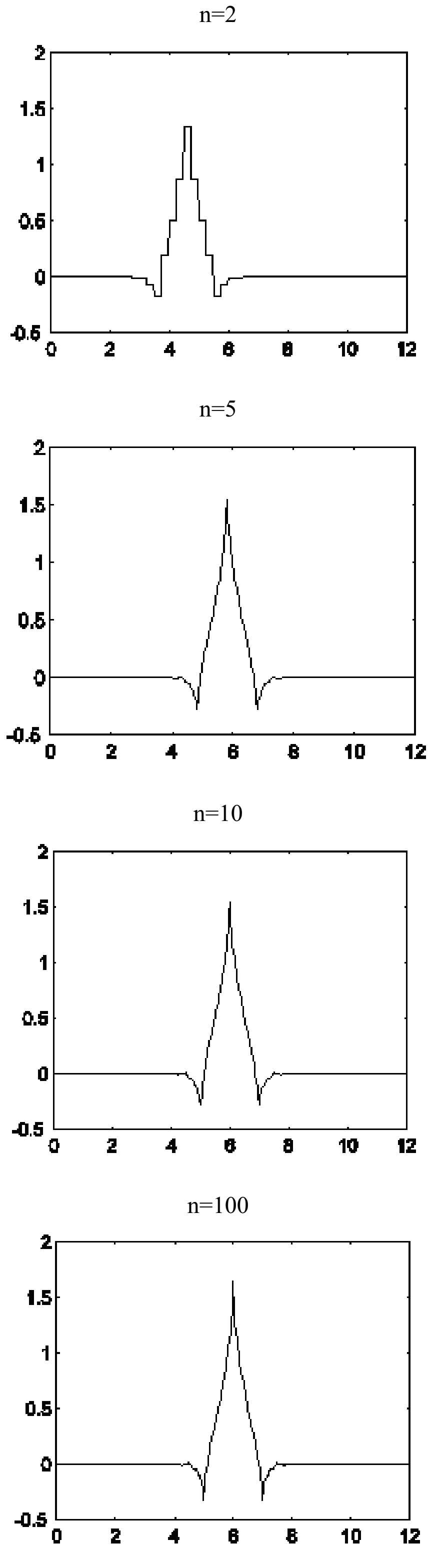

Figure 3. Convergence to scaling function for 1D case: m=1, M=5

From the process we can see that it convergences. (The scaling function obtained from 100 times recursion has a difference less than 6.639e-14 difference with the exact scaling function computed as in [24]).

The followed question is what is the wavelet corresponding to this scaling function, since the above filter bank is not perfect reconstructed. The wavelet corresponding to a scaling function is not unique, hence there are various alternatives. One way is construct a semi-orthogonal wavelet which always exists [21].

According to [21], once a scaling function is found, a semi-orthogonal wavelet always exists. It can be constructed as follows,

$$P_{so}(z) = P(z) \tag{26}$$

$$G_{so}(z) = P(z^{-1})E(z)/E(z^2) \tag{27}$$

$$Q_{so}(z) = -z^{-1}E(-z)P(z^{-1}) \tag{28}$$

$$H_{so}(z) = -zP(-z)/E(z^2) \tag{29}$$

Where $P_{so}(z)$, $Q_{so}(z)$, $G_{so}(z)$, $H_{so}(z)$ are the corresponding synthesis LPF, synthesis HPF, analysis LPF, analysis HPF, respectively, and $E(z)$ is a polynomial, which is defined as the z-transform of the correlation coefficients

$$E(z) = \sum_{k=-\infty}^{\infty} r_k z^{-k} \tag{30}$$

where $r(k)$ is correlation of the integer translates of the scaling function [21], i.e.,

$$r(y) = \int_{-\infty}^{\infty} \varphi(t)\overline{\varphi(t+y)}dt \tag{31}$$

Since φ is a real function, $r(k) = r(-k)$ and $E(z)$ is symmetric, i.e.,

$$E(z) = E(z^{-1}) \tag{32}$$

With the two-scale equation, a semi-orthogonal wavelet can be produced. The scaling function and its corresponding semi-orthogonal wavelet are shown in Fig. 4 (a) and (b) respectively for the case of m=2, M=5.

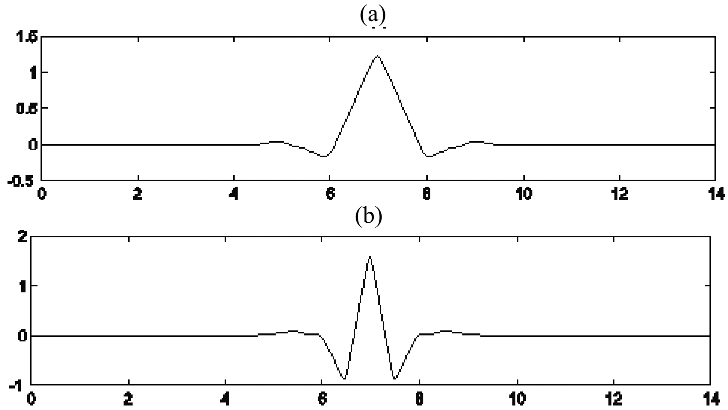

Figure 4. The 1D scaling function and semi-orthogonal wavelet as well as the biorthogonal wavelet for m=2, M=5

IV. CORRELATION ANALYSIS OF WAVELET FILTER BANKS

A. Correlation Analysis for the Scaling Function of the Semi-Orthogonal WaveletFfilter Bank

Another question is that what is the degree of orthogonality of these nearly-orthogonal wavelet filter banks, including the correlation among the integer translates of the scaling function, correlation of among the integer translates of the wavelet, and correlation between the integer translates of the scaling function and the integer translates of wavelet, etc. In a multiresolution analysis with an orthogonal wavelet basis, the integer translates of the scaling function ϕ from an orthogonal basis, the integer translates of the wavelet ψ from an orthogonal basis, the integer translates of the scaling function are orthogonal to the integer translates of the wavelet, and hence the integer translates of the wavelet at different scale are orthogonal each other, i.e.,

$$< \phi(t), \phi(t+k) >= \delta(k)$$

$$< \varphi(t), \varphi(t+k) >= \delta(k)$$

$$< \phi(t), \varphi(t+k) >= 0$$

Instead of study a single correlation coefficient $r_{\alpha,\beta}(k) =< \alpha(t), \beta(t+k) >$, it would be better to consider the correlation polynomial of the these coefficients

$$E_{\alpha,\beta}(z) = \sum_{k=-\infty}^{k=\infty} r_{\alpha,\beta}(k) z^{-k} \tag{33}$$

Therefore, we have

$$E_{\phi,\phi}(e^{j\omega}) = 1$$

$$E_{\psi,\psi}(e^{j\omega}) = 1$$

$$E_{\phi,\psi}(e^{j\omega}) = 0$$

Using Poisson summation formula, we have

$$E_{\alpha,\beta}(e^{j\omega}) = \sum_{k=-\infty}^{k=\infty} \hat{\alpha}(\omega + 2k\pi) \overline{\hat{\beta}(\omega + 2k\pi)} \tag{34}$$

We will use this relation in the following context.

If

$$| E_{\alpha,\beta}(e^{j\omega}) - 1 | < \varepsilon \tag{35}$$

we have

$$| r_{\alpha,\beta}(k) - \delta(k) | = | \frac{1}{2\pi} \int_0^{2\pi} E_{\alpha,\beta}(e^{j\omega}) e^{jk\omega} d\omega + \frac{1}{2\pi} \int_0^{2\pi} e^{jk\omega} d\omega |$$

$$\leq \frac{1}{2\pi} | \int_0^{2\pi} (E_{\alpha,\beta}(e^{j\omega}) - 1) e^{jk\omega} d\omega | \leq \varepsilon \tag{36}$$

which implies that

$1 - \varepsilon \leq r_{\alpha,\beta}(0) \leq 1 + \varepsilon$ and

$-\varepsilon \leq r_{\alpha,\beta}(k) \leq \varepsilon, \forall k \neq 0, k \in Z$. If

$$| E_{\alpha,\beta}(e^{j\omega}) | < \varepsilon \tag{37}$$

We would have

$$| r_{\alpha,\beta}(k) | = \frac{1}{2\pi} | \int_0^{2\pi} E_{\alpha,\beta}(e^{j\omega}) e^{jk\omega} d\omega | \tag{38}$$

$$\leq \varepsilon \tag{39}$$

which implies that $-\varepsilon \leq r_{\alpha,\beta}(k) \leq \varepsilon, \forall k \in Z$. From these relation, we see that using $E_{\alpha,\beta}(e^{j\omega})$ is able to describe the correlation of $\{\alpha(t+k), k \in Z\}$ and $\{\beta(t+k), k \in Z\}$. We use its difference to 1 or 0 as new criteria to measure the correlation properties.

Now, we consider the correlation of the integer translates of the scaling function, i.e., the series $\{\phi(t+k), k \in Z\}$. According to (30) and (34), we have

$$E(e^{j\omega}) = \sum_{k=-\infty}^{\infty} | \hat{\phi}(\omega + 2k\pi) |^2 \tag{40}$$

An important relation of $E(z)$ to $P(z)$ can be obtained similarly to that in [21]: (41)

$$2E(z^2) = P(z)P(z^{-1})E(z) + P(-z)P(-z^{-1})E(-z), |z| = 1$$

Therefore, it is easy to verify that the semi-orthogonal filter bank is perfect reconstructed. Note that [23],

$$\hat{\phi}(0) = 1; \hat{\phi}(2k\pi) = 0; k \neq 0 \tag{42}$$

Hence

$$E(1) = 1 \tag{43}$$

Since the length of $P(z)$ is finite, therefore the support of scaling function $\phi(t)$ doesn't exceed (-N, N) and $r(k) = 0, \forall k > N, k < -N$ and $E(z)$ can written as

$$E(z) = \sum_{k=12N}^{2N} r_k z^{-k} \tag{44}$$

Now let (45)

$$D(z) = P(z)P(z^{-1}) = d_{2N} z^{-2N} + \cdots + d_0 + \cdots + d_{2N} z^{2N}$$

If $P(z)$ is orthogonal, we have $d_0 = 1$ and $d_{2i} = 0$. For nearly orthogonal filter banks, we have

$$d_0 \approx 1, d_{2i} \approx 0 \tag{46}$$

And we have

$$E(z)D(z) + E(-z)D(-z) = 2E(z^2) \tag{47}$$

which can written as

$$e^T BZ + e^T BW_1 Z = 2e^T W_2 Z \tag{48}$$

where

$$e = [r_{2N}, \cdots, r_1, r_0, r_1, \cdots, r_{2N}]$$

$$Z = [z^{-4N}, \cdots, 1, \cdots, z^{4N}]$$

$$B = \begin{pmatrix} d_{2N} & \cdots & d_1 & d_0 & d_1 & \cdots & d_{2N} & & & \\ & d_{2N} & \cdots & d_1 & d_0 & d_1 & \cdots & d_{2N} & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & d_{2N} & \cdots & d_1 & d_0 & d_1 & \cdots & d_{2N} \end{pmatrix}$$

$$W_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & & & & & & & \\ 0 & 0 & 1 & 0 & 0 & & & & & & & \\ 0 & 0 & 0 & 0 & 1 & & & & & & & \\ & & & & & \ddots & & & & & & \\ & & & & & & 1 & & & & & \\ & & & & & & & \ddots & & & & \\ & & & & & & & & 1 & 0 & 0 & 0 & 0 \\ & & & & & & & & 0 & 0 & 1 & 0 & 0 \\ & & & & & & & & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

is a (4N+1) × (8N+1) matrix and W_1 is a (8N+1) × (8N+1) diagonal matrix with all entries being zeros, except on the diagonal the entries are alternatively 1 and -1 with the central entry being 1. Hence

$$((I + W_1)B^T - 2W_2^T)e = 0 \tag{49}$$

Notice that the even columns of the above matrix are zero, remove these columns, we have

$$\overline{B}e = 0 \tag{50}$$

where $\overline{B}$ is a matrix whose columns are denoted as

$$\overline{B} = [a_{2N}, a_{2N-1}, \cdots, a_1, a_0, \overline{a_1}, \cdots, \overline{a_{2N-1}}, \overline{a_{2N}}] \tag{51}$$

where

$$a_{2N} = [d_{2N} - 1, d_{2N-2}, \cdots, d_{2N-2}, d_{2N}, 0, \cdots, 0]^T$$
$$a_{2N-1} = [0, d_{2N-1} - 1, d_{2N-3}, \cdots, d_{2N-3}, d_{2N-1}, 0, \cdots, 0]^T$$
$$a_{2N-2i} = [\underbrace{0, \cdots, 0}_{i}, d_{2N}, d_{2N-2}, \cdots, d_{2N-2i} - 1, \cdots, d_{2N-2}, d_{2N}, 0, \cdots, 0]^T$$
$$a_{2N-1-2i} = [\underbrace{0, \cdots, 0}_{i+1}, d_{2N-1}, d_{2N-3}, \cdots, d_{2N-1-2i} - 1, \cdots, d_{2N-3}, d_{2N-1}, 0, \cdots, 0]^T$$

where all vectors are of length of 4N+1, i is take from 1 to the integer such that 2N-2i and 2N-1-2i are nonnegative, $\overline{a_i}$ is a vector by flipping the vector of a_i upside down, i.e., $\overline{a_i} = Ja_i$, and J is an anti-diagonal matrix with entries on the anti-diagonal is 1. It is worthwhile to notice that (52)

$$a_0 = [0, \cdots, 0, d_{2N}, d_{2N-2}, \cdots, d_0 - 1, \cdots, d_{2N-2}, d_{2N}, 0, \cdots, 0]^T$$

which is almost equal to zero. Now, we have

$$A\bar{e} = a_0 r_0 \tag{53}$$

where

$$\bar{e} = [r_{2N}, \cdots, r_1]^T \tag{54}$$

$$A = (I + J)[a_{2N}, \cdots, a_1] \tag{55}$$

Hence, we have

$$\bar{e} = (A^T A)^{-1} A^T r_0 a_0 \tag{56}$$

Equation (31) implies that

$$r_0 + 2\sum_{k=1}^{2N} r_i = 1 \tag{57}$$

Hence, we have

$$r_0 = \frac{1}{1 + 2b(A^T A)^{-1} A^T a_0} \tag{58}$$

where b=[1,1,⋯,1]. Since every element $a_0 \approx 0$ is, we can see from the above equation that $r_0 \approx 1, r(i) \approx 0, \forall i \neq 0$, and $E(e^{j\omega}) \approx 1$ [26]. Hence the integer translates of the scaling function is nearly orthogonal. For $m = 1 \sim 3, n = 2 \sim 5$, we have computed the $E(e^{j\omega})$, and its the maximum difference to 1 in decibel, $20\log(|E(e^{j\omega}) - 1|)$, is given in Table 2. For the case of m=2, M=5, the correlation $r(i), k \in Z$ are shown in Fig. 5. We see that the integer translates of the scaling function are almost orthogonal.

TABLE 2

AMPLITUDE DISTORTION OF THE DUAL NEARLY ORTHOGONAL FILTER BANK

	m=1	m=2	m=3
n=2	72.03	50.96	21.71
n=3	101.51	53.34	50.44
n=4	114.67	46.37	60.83
n=5	118.80	88.28	44.69

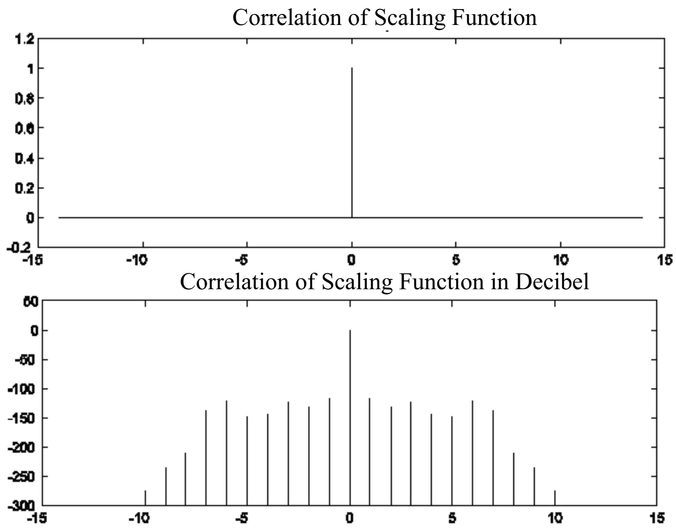

Figure 5. Correlation coefficients of the scaling function [Trial mode] in decibel

B. Correlation Analysis for the Semi-Orthogonal Wavelet function

Now let us consider the correlation of wavelet in the semi-orthogonal wavelet filter bank, which can be obtained via the two-scaling equation

$$\hat{\psi}(\omega)=\frac{1}{\sqrt{2}}Q_{so}(e^{j\omega})\hat{\phi}(\omega/2) \qquad (59)$$

Similar to the correlation analysis of the scaling function, we consider (60)

$$2E_{\psi}(e^{j\omega})=2\sum_k r_{\psi}(k)e^{jk\omega}=2\sum_{k=-\infty}^{\infty}|\hat{\psi}(\omega+2k\pi)|^2$$

$$=\sum_{k=-\infty}^{\infty}|Q_{so}(e^{j(\omega+2k\pi)/2})|^2|\hat{\psi}(\omega+2k\pi)/2|^2$$

$$=\sum_{k=-\infty}^{\infty}\left|Q_{so}\left(e^{j(\omega/2+2k\pi)}\right)\right|^2\left|\hat{\psi}\left(\omega/2+2k\pi\right)\right|^2$$

$$+\sum_{k=-\infty}^{\infty}\left|Q_{so}\left(e^{j(\omega/2+\pi+2k\pi)}\right)\right|^2\left|\hat{\psi}\left(\omega/2+\pi+2k\pi\right)\right|^2$$

$$=|Q_{so}(e^{j\omega/2})|^2\ E(e^{j\omega/2})+|Q_{so}(e^{j(\omega/2+\pi)})|^2\ E(e^{j(\omega/2+\pi)})$$

Using the definition of (28) and the relation (41), we have

$$2E_{\psi}\left(e^{j\omega}\right)$$

$$=E\left(-e^{j\omega/2}\right)E\left(-e^{-j\omega/2}\right)P\left(-e^{j\omega/2}\right)P\left(-e^{-j\omega/2}\right)E\left(e^{-j\omega/2}\right)$$

$$+E(e^{j\omega/2})E(e^{-j\omega/2})P(e^{j\omega/2})P(e^{-j\omega/2})E(-e^{-j\omega/2})$$

$$=E\left(-e^{j\omega/2}\right)E\left(-e^{-j\omega/2}\right)\Big[P\left(-e^{j\omega/2}\right)P\left(-e^{-j\omega/2}\right)$$

$$\cdot E\left(e^{-j\omega/2}\right)+P\left(e^{j\omega/2}\right)P\left(e^{-j\omega/2}\right)E\left(e^{-j\omega/2}\right)\Big]$$

$$=2E(-e^{j\omega/2})E(-e^{-j\omega/2})E(e^{j\omega}) \qquad (61)$$

Hence we obtain

$$E_{\psi}(e^{j\omega})=E(-e^{j\omega/2})E(e^{j\omega/2})E(e^{j\omega}) \qquad (62)$$

For example, the correlation coefficients of the semi-orthogonal wavelet for the case of m=2, M=5 are shown in Fig.6. We see that the correlation is very small.

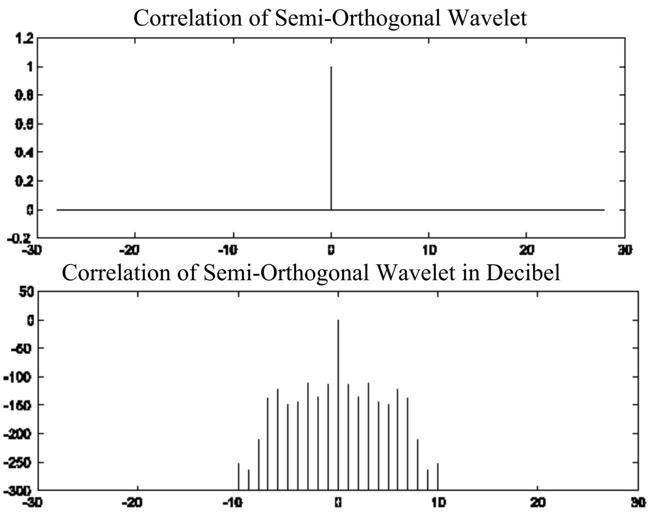

Figure 6. Correlation coefficients of the semi-orthogonal wavelet [Trial mode] in decibel

If we assume that the bound for the distortion of $E(e^{j\omega})$ to 1 is

$$|E(e^{j\omega})-1|\le\varepsilon \qquad (63)$$

then we have

$$(1-\varepsilon)^3\le E_{\psi}(e^{j\omega})\le(1+\varepsilon)^3 \qquad (64)$$

Now, consider the correlation of integer translates of the wavelet and the scaling function, and using (34), we have

$$E_{\phi,\psi}(e^{j\omega})=\sum_{k=-\infty}^{\infty}\hat{\phi}(\omega+2k\pi)\,\overline{\hat{\phi}(\omega+2k\pi)}$$

$$=\sum_{k=-\infty}^{\infty}P_{so}\left(e^{j(\omega/2+k\pi)}\right)\hat{\phi}\left(\omega/2+k\pi\right)$$

$$\cdot Q_{so}\left(e^{j(\omega/2+k\pi)}\right)\hat{\phi}\left(\omega/2+k\pi\right)$$

$$=\sum_{k=-\infty}^{\infty}P_{so}(e^{j(\omega/2+k\pi)})Q_{so}(e^{-j(\omega/2+k\pi)})|\hat{\phi}(\omega/2+k\pi)|^2$$

$$=P_{so}\left(e^{j\omega/2}\right)Q_{so}\left(e^{-j\omega/2}\right)E\left(e^{j\omega/2}\right)$$

$$+P_{so}\left(-e^{j\omega/2}\right)Q_{so}\left(-e^{-j\omega/2}\right)E\left(-e^{j\omega/2}\right) \qquad (65)$$

Notice that

$$P_{so}(e^{j\omega})=P(e^{j\omega}),Q_{so}(e^{-j\omega})=-e^{j\omega}E(-e^{j\omega})P(-e^{-j\omega})$$

from (26) and (28), we have

$$E_{\phi,\psi}(e^{j\omega})$$

$$=P(e^{j\omega/2})(-e^{-j\omega/2})E(-e^{-j\omega/2})P(-e^{j\omega/2})E(e^{j\omega/2})$$

$$+P(-e^{j\omega/2})e^{-j\omega/2}E(e^{-j\omega/2})P(e^{j\omega/2})E(-e^{j\omega/2})$$

$$=0 \qquad (66)$$

This indicates that the wavelet at different scale is orthogonal. Because $\psi(t/2)=\sqrt{2}\sum_k q(k)\phi(t-k)$ is orthogonal to $\psi(t-k)$.

Consider the correlation of the dual scaling function and the dual wavelet, we can show that they are also nearly orthogonal. Notice that $G_{so}(z)=E(z)P(z^{-1})/E(z^2)$ also has four zeros at $z=-1$ as the filter $P(z)$, the corresponding scaling function $\tilde{\phi}(t)$ can be derived as

$$\hat{\tilde{\phi}}(\omega) = \lim_{n\to\infty}\prod_{k=1}^{n}\frac{1}{\sqrt{2}}G_{so}(e^{j\omega/2^k})$$

$$= \lim_{n\to\infty}\prod_{k=1}^{n}\frac{1}{\sqrt{2}}\frac{E(e^{j\omega/2^k})P(e^{-j\omega/2^k})}{E(e^{j\omega/2^{k-1}})}$$

$$= \lim_{n\to\infty}\frac{E(e^{j\omega/2^n})}{E(e^{j\omega})}\prod_{k=1}^{n}\frac{1}{\sqrt{2}}P(e^{-j\omega/2^k})$$

$$= \frac{E(1)}{E(e^{j\omega})}\hat{\phi}(-\omega) = \frac{1}{E(e^{j\omega})}\hat{\phi}(-\omega) \quad (67)$$

Therefore, the correlation polynomial is equal to

$$E_{\tilde{\phi}}(e^{j\omega}) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}|\hat{\tilde{\phi}}(\omega+2k\pi)|^2$$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\frac{|\hat{\phi}(\omega+2k\pi)|^2}{E^2(e^{j\omega})}$$

$$= \frac{E(e^{-j\omega})}{E^2(e^{j\omega})} = \frac{1}{E(e^{j\omega})} \quad (68)$$

Using (62) holds, we have

$$\frac{1}{1+\varepsilon} \le E_{\tilde{\phi}}(e^{j\omega}) \le \frac{1}{1-\varepsilon} \quad (69)$$

The correlation of the dual wavelet can be obtained similarly to (60) as

$$E_{\tilde{\psi}}\left(e^{2\omega}\right) = H_{so}\left(e^{j\omega}\right)H_{so}\left(e^{-j\omega}\right)E_{\tilde{\phi}}\left(e^{j\omega}\right)$$

$$+H_{so}\left(-e^{j\omega}\right)H_{so}\left(-e^{-j\omega}\right)E_{\tilde{\phi}}\left(-Se^{j\omega}\right)$$

$$= \frac{-e^{j\omega}P\left(-e^{j\omega}\right)\left(-e^{j\omega}\right)P\left(-e^{-j\omega}\right)}{E\left(e^{2j\omega}\right)E\left(e^{2j\omega}\right)E\left(e^{j\omega}\right)}$$

$$+\frac{e^{j\omega}P\left(e^{j\omega}\right)\left(e^{-j\omega}\right)P\left(e^{-j\omega}\right)}{E\left(e^{2j\omega}\right)E\left(e^{2j\omega}\right)E\left(-e^{j\omega}\right)}$$

$$= \frac{1}{E(e^{2\omega})E(e^{j\omega})E(-e^{j\omega})} \quad (70)$$

The bound for $E_{\tilde{\psi}}(z)$ is

$$\frac{1}{(1+\varepsilon)^3} \le E_{\tilde{\phi}}(e^{j\omega}) \le \frac{1}{(1-\varepsilon)^3} \quad (71)$$

Similarly, we can show that the integer translates of the dual scaling function and the integer translates of the dual wavelet are orthogonal.

From the above analysis, we see that the semi-orthogonal wavelet is nearly-orthogonal. Moreover, this filter bank is very close to the nearly-orthogonal filter bank. Since $E(z)$ has zero phase, all the filters in the semi-orthogonal filter bank have the same phase response as those in the nearly-orthogonal filter bank, and they have almost the same amplitude response as those in the nearly-orthogonal filter bank. Let

$$C_p = \max|P(e^{j\omega})| \quad (72)$$

be the maximum value of the amplitude response of $P(z)$. Then we have the following bound for the amplitude difference between the corresponding filters in the nearly-orthogonal filter bank and the semi-orthogonal wavelet filter bank

$$|P_{so}(e^{j\omega}) - P_{no}(e^{j\omega})| = 0$$

$$|Q_{so}(e^{j\omega}) - Q_{no}(e^{j\omega})| \le C_p\varepsilon$$

$$|G_{so}(e^{j\omega}) - G_{no}(e^{j\omega})| \le \frac{2C_p\varepsilon}{1-\varepsilon}$$

$$|H_{so}(e^{j\omega}) - H_{no}(e^{j\omega})| \le \frac{C_p\varepsilon}{1-\varepsilon}$$

Therefore, if ε is small enough, the semi-orthogonal filter bank can be implemented as the nearly-orthogonal filter, and the latter can be regarded as an approximate implementation of the corresponding semi-orthogonal filter bank.

V. CONCLUSIONS

The nearly orthogonal one-dimensional filter banks are designed and analyzed in this paper. Especially, the orthogonality and correlation of the wavelet bases are studied. The main idea is to optimize a symmetric filter with multiple zeros at the aliasing frequency to satisfy the orthogonal condition, then a nearly-orthogonal filter bank can be obtained. Design examples with small even number of zeros at the aliasing frequencies and short length filter banks are given.

Since the nearly-orthogonal filter bank is not perfect reconstructed, to analyze the properties of the wavelet, we need to construct perfect reconstructed filter banks. Semi-orthogonal filter banks are constructed, and detailed correlation analysis made on the semi-orthogonal filter banks show that it is nearly-orthogonal. The integer translates of the wavelet and scaling functions are nearly-orthogonal. The integer translates of the wavelet at different scale are completely orthogonal. Theoretical analysis and experiments show that these filter banks are very close to the nearly-orthogonal filter bank if the filter bank is very close to orthogonal. It can be implemented as the corresponding nearly- orthogonal filter bank, in which every filter in the nearly-orthogonal filter bank is a very good approximation of the corresponding filter in the semi-orthogonal filter bank.

ACKNOWLEDGMENTS

The authors would like to thank Dr. Chunyu Yu and Dr. Zejun Wu for their contributions. Special thanks go to Prof. Jianshi Li for his advice and patience.

REFERENCES

[1] M. Vetterli and J. Kovacevic, "Wavelets and Subband Coding", Prentice Hall PTR, Englewood Cliffs, New Jersey, 1995.

[2] P.P. Vaidyanathan, "Multirate Systems and Filter Banks", Prentice Hall PTR, Englewood Cliffs, New Jersey, 1993.

[3] M. Antonini, M. Barlaud, P. Mathieu and I. Daubechies, "Image coding using wavelet transform", IEEE Trans. Image Processing, vol.1, no.2, pp.205-220, April 1992.

[4] J.M. Shapiro, "Embedded Image Coding Using Zerotrees of Wavelet Coefficients", IEEE Trans. Signal Processing, vol. 41, no.12, pp.3445-3462, 1993.

[5] J.D. Villasenor, B. Belzer and J. Liao, "Wavelet Filter Evaluation for image compression", IEEE Trans. Image Processing, vol. 4, no. 8, pp.1053-1060, 1995.

[6] E.A.B. da Silva and M. Ghanbari, "On the performance of linear phase wavelet transforms in low bit-rate image coding", IEEE Trans. Image Processing, vol.5, no.5, pp.689-704, May 1996.

[7] P. Sriram and M.W. Maicellin, "Image Coding Using Wavelet Transforms and Entropy-Constrained Trellis-Coded Quantization", IEEE Trans. Image Processing, vol.4, no.6, pp.725-733, June 1995.

[8] Z. C. Bao, X. G. You, C. F. Xing and Q. Y. He, "Image Denoising by using Non-Tensor Product Wavelets Filter Banks", 2007 International Conference on Machine Learning and Cybernetics, vol. 3, pp. 1734–1738, Aug. 2007.

[9] J. Barros and R. I. Diego, "Analysis of Harmonics in Power Systems Using the Wavelet-Packet Transform", IEEE Trans. Instrumentation and Measurement, vol.57, no.1, pp.63–69, Jan. 2008.

[10] T. Chen and P. Vaidyanathan, "Multidimensional multirate filters and filter banks derived from one-dimensional filters", IEEE Trans. Signal Processing, Vol. 41, No.5, pp. 1749-1765, 1993.

[11] S. Yao, "Signal-adapted wavelet filter bank design", Electronics Letters, vol.34, no.5, pp.434–436, March 1998.

[12] G. Karlsson and M. Vetterli, "Theory of two-dimensional multirate filter banks", IEEE Trans. Acoust., Speech, Signal Processing, vol.38, no.6, pp.925-937, 1990.

[13] R.E. Crochiere and R. L. Rabiner, Multirate Digital Signal Processing, Prentice-Hall, Englewood Cliffs, NJ, 1983.

[14] S. T. Hsiang, J. W. Woods and J. R. Ohm, "Invertible temporal subband/wavelet filter banks with half-pixel-accurate motion compensation", IEEE. Trans. Image Processing, vol.13, no.8, pp.1018-1028, Aug. 2004.

[15] A. M. Jalil, H. Amindavar and F. Almasganj, "Subband blind equalization using wavelet filter banks", IEEE International Symposium on Circuits and Systems, Vol. 6, pp.5730-5733, May 2005.

[16] S. K. Kwon and J. K. Kim, "New regular M-band orthogonal wavelet filter bank design using zero-insertion method", Electronics Letters, vol.30, no.10, pp. 753-754, May 1994.

[17] Y. Zhao and M. N. S. Swamy, "New technique for designing nearly orthogonal wavelet filter banks with linear phase", IEE Proceedings - Vision, Image and Signal Processing, vol.147, no.6, pp.527-533, Dec. 2000.

[18] X. Zhang, W. Wang, T. Yoshikawa and Y. Takei, "Lifting implementation of IIR orthogonal wavelet filter banks using allpass filters", The 47th Midwest Symposium on Circuits and Systems, vol.2, pp.413-416, July 2004.

[19] E.H. Adelson, E. Simoncelli and R. Hingorani, " Orthogonal pyramid transform for image coding", In Proc. SPIE, vol. 845, pp.50-58, Cambridge, MA, October 1987.

[20] A. Srinivasarao, R. Satyabama and S. Annadurai, "Implementation of Adaptive Lifting Scheme in the Design of IIR Orthogonal Wavelet Filter Banks", The International Conference on Computational Intelligence and Multimedia Applications, vol.3, pp.324-332, Dec. 2007.

[21] C.K. Chui, An Introduction to Wavelets, Academic Press, New York, 1992.

[22] D. Sersic, "Wavelet filter banks with adaptive number of zero moments", The 5th International Conference on Signal Processing, vol.1, pp.325-328, Aug. 2000.

[23] D. Sersic, "Integer to integer mapping wavelet filter bank with adaptive number of zero moments", The International Conference on Acoustics, Speech, and Signal Processing, vol.1, pp.480-483, June 2000.

[24] S. Mallat, "A theory for multiresolution signal decomposition: the wavelet representation", IEEE Trans. Patt. Anal. Machine Intell., vol.11, no.7, pp.674-693, 1989.

[25] G. Strang, "Wavelet and dilation equation", SIAM Journal of Math. Analy., vol .31, pp.614-627, 1989.

[26] Y. Zhao and M.N.S. Swamy, "A Technique for Designing New Biorthogonal Filters and Its Application to Image Compression", Electronics Letters, vol. 35, no.18, 1999.

Yun Wu was born in Jingdezhen, China, in 1973. He received the B.E. degree in computer science from Nanjing University of Aeronautics and Astronautics (NUAA), Nanjing, China, in 1999, and the M.E. degree in computer science from Guizhou University, Guiyang, China, in 2006. As a doctoral student who is joint-cultured by Peking University and Guizhou University, he has been working toward his PhD degree in computer science since 2006.

From 2003 to 2006, he was with Guizhou University, Guiyang, China, where he was engaged in research work on image processing and its software developments. Since 2006, he has been with Shenzhen Graduate School of Peking University, Shenzhen, China, where he is currently engaged in research work on video coding. His current interests are video coding, wavelet transform, and digital signal processing.

Mr. Wu received the prize of the Graduate Fellowships of Guizhou University in 2004 and the Graduate Research Fund Award of Guizhou University in 2008.

Yong Zhao was born in Changshun, China, in 1963. He received the B.E. degree in mathematics from Guizhou University, Guiyang, China, in 1985, and the M.E. degree from Department of Information Science, Northwestern Polytechnic University, Xi'an, China, in 1988, and the PhD degree in Engineering from Research Institute of Automation, Southeast University, Nanjing, China, in 1991.

From 1991 to 1994, he worked as a research assistant in Dept. of Biomedical Engineering, Zhejiang University. From 1994 to 1997, he worked as a vice dean/associate professor in Dept. of Electrical Engineering, Hangzhou University (Now the Zhejiang University), independently designed the whole hardware and software of Pocket Human Movement Amount Monitor. From 1997 to 2000, he worked as a post-doctoral fellow in Center for Signal Processing and Communications, Electrical and Computer Engineering, Concordia University, Montreal, Canada. From 2000 to 2004, he worked as a senior software engineer in Honeywell Corporation, Ottawa, Canada. Since 2004, he has been with Shenzhen Graduate School of Peking University, Shenzhen, China, where he is currently a Professor. He has published more than 50 papers in journals and conferences and also a book on DSP-based video application. He has 5 invention patents and 1 practical patent. His main research interests are video compression and video intelligent analysis, as well as their applications to commercial products, including: IP camera, video servers, video surveillance system, intelligent video surveillance, and video analysis based vehicle safety, etc.

Dr. Zhao is an executive member of the circuits and systems committee of Zhejiang Province and a Chinese Institute of Electronics Senior Member. He Won support from National Natural Science Foundation in 1996, received the second prize of the Science and Technology Progress Award of Shenzhen in 2007.

Ling Shen was born in Meishan, China, in 1970. He received the B.E. degree in Engineering from Department of Electrical Engineering, Sichuan University, Chengdu, China, in 1992, and the M.E. degree from Institute of Acoustics, Chinese Academy of Sciences, Beijing, China, in 1995, and the PhD degree in Engineering from Department of Electrical and Computer Engineering, National University of Singapore, in 2005.

From 1995 to 1997, he worked as a research assistant in Institute of Acoustics, participated in the developing of a human-machine speech communication system. From 2000 to 2003, he developed Chinese and Malay text-to-speech system while working as a speech system designer in Singapore Mindmaker Pte., Ltd. From 2006 to 2007, he developed algorithms for LDPC Decoder for DMB DTV demodulator in Shenzhen Leaguerme Microelectronics Co., Ltd. Since 2007, he has been engaged in teaching and research as a lecturer in Shenzhen Graduate School of Peking University. Research interests include: audio-video codec, forward error correction, artificial intelligence, speech and music signal processing.