Analytical Valuation of Contingent Claims by Stochastic Interacting Systems for Stock Market

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Abstract—In the present paper, by applying the theory of stochastic processes and interacting particle systems and models, including stopping time theory and stochastic voter model, we model a financial stock price model that contains two types of investors, and we use this financial model to describe the behavior and fluctuations of a stock price process in a stock market. In the financial model, besides the professional investors, we also consider the general investors or nonprofessional investors, where the stopping time and the voter model are applied to model and study the statistical properties of investment of the nonprofessional investors. By using the stochastic methods of statistical analysis, we show that the probability distribution of the normalized random price process for this financial model converges to the corresponding distribution of the Black-Scholes model. Further, we discuss the valuation and hedging of European contingent claims for this price process model.

Index Terms—interacting particle systems, stochastic processes, stock price model, fluctuation, European contingent claims, valuation, hedging

I. INTRODUCTION

The study of fluctuation of stock price has been made great progress in the past ten years, see [5-9,13,14,16,17]. Recently, some research work has been done in the field of applying the theory of stochastic interacting particle dynamic systems to investigate the statistical properties of fluctuations of stock prices in a stock market, and the corresponding valuation and hedging of contingent claims for this price process model are also studied, for example see [6,7,16,17]. As the stock markets are becoming deregulated worldwide, the modeling of the dynamics of the forwards prices is becoming a key problem in the risk management, physical assets valuation, and derivatives pricing, and it is also important to understand the statistical properties of fluctuations of stock price in globalized securities markets, and it is useful for valuation and hedging of European option and American option.

In the present paper, the interacting particle dynamic systems and stopping time theory are used to construct a financial price model, the price model is a non-Markovian process approach. In a stock market, the investors usually consist of two types of investors, type-1 investors are the professional investors, type-2 investors are the general investors or nonprofessional investors. So the financial model of this paper contains two types of investors, and a particle in the model is considered as an investor or a trader, the interface of the model is considered as a stock price process, see [17]. Further, we discuss the valuation and hedging of European contingent claims for this price model.

II. MODELING THE STOCK PRICES BY THE INTERACTING PARTICLE SYSTEMS

First we give the construction of this financial model. For a stock market, we consider two types of market participants or investors, type-1 market participants are called the professional investors, they decide their trading positions by analyzing the government’s investment policy, the past market data, the trading strategy, investment risk, returns and so on. Type-2 market participants are called the general investors or nonprofessional investors, they usually decide their trading positions by news which is randomly obtained or obtained by other investors.

For type-1 investors, we consider a single stock, and assume that there are $m_i$ (large enough) traders in this stock, and each trader can trade unit number of stocks at each time $t$. At time $t$, the behavior of stock price process is partially determined by the number of traders $a_i^+$ (with buying positions) and $a_i^-$ (with selling positions). Let ‘$+$’, ‘$-$’ and ‘$0$’ denote that the traders take buying positions, selling positions and neutral positions respectively. If the number of traders in buying positions is bigger than the number of traders in selling positions, it implies that the stock price is considered to be low by the market participants, and the stock price auctions higher searching for buyers, similarly for the...
opposite case. Let \( \omega_t(m) \) be the position of trader \( m \leq m \leq m_t \) at time \( t(t=1,2,\cdots) \), and \( \omega_t = \{ \omega_1(m), \cdots, \omega_t(m) \} \) be the configuration of positions for \( m_t \) traders. A space of all configurations of positions for \( m_t \) traders from time 1 to \( n(t=1,\cdots,n) \) is given by \( \Omega = \{ \omega \} \). For a given configuration \( \omega = \{ \omega_1, \cdots, \omega_n \} \in \Omega \) and \( t=1,\cdots,n \), let

\[
A(\omega) = \begin{cases} 
\omega_i^+ - \omega_i^- - h_0 & \text{if } \omega_i^+ - \omega_i^- > h_0 \\
0 & \text{if } |\omega_i^+ - \omega_i^-| \leq h_0 \\
-(\omega_i^+ - \omega_i^- - h_0) & \text{if } \omega_i^+ - \omega_i^- > h_0 
\end{cases} (1)
\]

where \( h_0 \) is a positive market parameter. If \( A(\omega_t) > 0 \), there are more buyers than sellers and the stock price is auctioned up. From above definitions and [8,15,19], we define the stock price of type-1 at time \( t=1,\cdots,n \) as following \( S^t_{\omega_0} = e^{A(\omega_t)} S^t_{\omega_0} \), where \( \omega_0 > 0 \), and let \( S_0 \) be the initial price at time \( t = 0 \). Then we have

\[
S^t_{\omega_0} = S_0 \exp \{ A(t) \} . (2)
\]

The trade volume of this stock at time \( t \) is given by \( v(\omega) = \min \{ \omega_i^+, \omega_i^- \} \). Then, for a given configuration \( \omega \in \Omega \), we say \( \omega_i \) is static if \( \omega_i^+ + \omega_i^- \leq h_0 \), and active if \( \omega_i^+ + \omega_i^- > h_0 \). Let

\[
\omega_i^+ = \begin{cases} 
\omega_i^+ + \omega_i^- - h_0 & \text{if } \omega_i \text{ is active} \\
0 & \text{otherwise} 
\end{cases} (3)
\]

denote the number which corresponds to the market participants. Next we give a Gibbs measure on the configuration space \( \Omega \). Let \( H(\omega) \) be the Hamiltonian (see [3,10])

\[
H(\omega) = \beta \left( \sum_{i=1}^{n} \omega_i^+ \right)^{\gamma} + \sum_{i=1}^{n} f(\omega_i,v(\omega)) + \delta \sum_{i=1}^{n} g(\omega) \quad (4)
\]

where \( \beta, \gamma \) and \( \delta \) are positive parameters and \( d \) is a positive constant. The first term of above equation denote the modified number of market participants, \( \beta \) controls the strength of the number of market participants, and we suppose that the number of market participants plays an important role for this price model. The second term is a function corresponding to the trade volume of the stock, and we include the market sentiment and trading strategies to the third term. The functions \( f(\omega_i,v(\omega)) \) and \( g(\omega) \) satisfy the following conditions: (I) if \( \omega_i \) is static, then \( f(\omega_i,v(\omega)) = g(\omega) = 0 \); (II) \( f(\omega_i,v(\omega)) \) and \( g(\omega) \) are symmetric for \( \omega_i \), that is, \( f(-\omega_i,v(-\omega_i)) = f(\omega_i,v(\omega)) \) and \( g(-\omega_i) = g(\omega) \); (III) there are some \( c^+ > 0, c^0 > 0 \), such that \( f(\omega_i,v(\omega)) \geq c^+ \omega_i^+ \) and \( g(\omega) \geq c^0 \omega_i^+ \). The Gibbs measure associated with the Hamiltonian \( H(\omega) \) is defined as

\[
P^t(\omega) = \frac{\exp[-H(\omega)]}{Z_n} \quad (5)
\]

where the configuration \( \omega \in \Omega \), and \( Z_n = \sum_{\omega \in \Omega} \exp[-H(\omega)] \) is a partition function.

In order to give a description of Hamiltonian \( H(\omega) \) and Gibbs measure \( P(\omega) \), we give an example in the following. Let \( d = 4 \), and \( h_1, h_2 \) be two positive constants such that \( h_2 > h_1 > h_1 \), we define

\[
f(\omega_i,v(\omega)) = -C_{1}(t)[v(\omega_i)]^2 \omega_i^+, \quad (6)
\]

and

\[
g(\omega) = \begin{cases} 
-C_{1}(t)[(\omega_i^+)^2 - (\omega_i^-)^2] \omega_i^+, & \text{if } \omega_i^+ > h_1 \\
0, & \text{if } 0 \leq \omega_i^+ \leq h_1 
\end{cases} (7)
\]

where \( \omega_0 = 0 > 0 \). For the function \( f(\omega_i,v(\omega)) \) of (6), the increasing of trade volume \( v(\omega) \) works for increasing the number of market participants, this plays an important role for the activity of the stock market. In (7), we give an example of market sentiment and trading strategies function \( g(\omega) \). If \( \omega_i^+ \in [0,h_1] \), then we have \( g(\omega) = 0 \), this means that the function \( g(\omega) \) has no influence on the investors’ positions; if \( \omega_i^+ \in (h_1,h_2) \), then we have \( g(\omega) < 0 \), the interaction function \( g(\omega) \) will increase the number of market participants; if \( \omega_i^+ \in (h_2,\infty) \), then \( g(\omega) > 0 \), it will decrease the number of market participants.

Next we discuss type-2 investors, they usually decide their trading positions by news, and their investment behavior is often described as the herd behavior. Here we apply interacting particle model --- the voter model, to study the fluctuations of price process which is determined by type-2 investors. Now, we give the brief definitions and properties of the voter model, for details see [10]. One interpretation for the voter model is, for a collection of individuals, each of which has one of two possible positions on a political issue, at independent exponential times, an individual reassesses his view by choosing a neighbor at random with certain probabilities and then adopting his position. Specifically, the voter model is one of the statistical physics models, we think of the sites of the \( d \)-dimensional integer lattice as being occupied by persons who either in favor of or opposed to some issue. To write this as a set--valued process, we let \( \{\xi(s), s \geq 0\} \) the set of voters in favor, we can also think of the sites in \( \xi(s) \) as being occupied by cancer cells, and the other sites as being occupied by healthy cells. We can formulate the dynamics as follows: (i) An occupied site becomes vacant at a rate equal to the number of the vacant neighbors; (ii) An vacant site becomes occupied at a rate equal to \( \lambda \) times the number of the occupied neighbors, where \( \lambda \) is a intensity which is called the “carcinogenic advantage” in voter model. When \( \lambda = 1 \),
the model is called the voter model, and when $\lambda > 1$, the model is called the biased voter model.

Let $\xi^d(s)$ ($s \in I$) denote the state at time $s$ with the initial state $\xi^d(0) = A$, then from [10], the voter model $\xi^d(s)$ approaches total consensus in $d = 1$ and $d = 2$. But in higher dimensions $d \geq 3$, the differences of opinion may persist. For the biased voter model ($\lambda > 1$), there is a "critical value" for the process, it can be shown that $\lambda_c = 1$ for the voter model. This means that, on $d$-dimensional lattice, if $\lambda < \lambda_c$, the process dies out (becomes vacant) exponentially fast, if $\lambda > \lambda_c$, the process survives with positive probability.

Consider a model of auctions for the same stock defined above, we can derive the stock price process from the auctions. Assume that each trader can trade the stock several times at each day $t \in \{1, 2, \cdots, n\}$, but at most one unit number of the stock at each time. Let $t$ be the time length of trading time in each trading day, we denote the stock price at time $s$ in the $t$-th trading day by $S_{i}^{type-2}(s)$, where $s \in [0, T]$. Suppose that this stock consists of $m(t) + 1$ ($m(t) + 1$ is large enough) investors, who are located in a line $\{-m(t)/2, \cdots, -1, 0, 1, \cdots, m(t)/2\}$ (similarly for $d$-dimensional lattice $\mathbb{Z}^d$). At the beginning of trading in each day, suppose that only the investor at the site 0 receives some news. We define a random variable $\text{sgn}(0)$ for this investor, suppose that this investor taking buying position ($\text{sgn}(0) = 1$), selling position ($\text{sgn}(0) = -1$) or neutral position ($\text{sgn}(0) = 0$) with probability $p_1, p_{-1}$ or $1 - (p_1 + p_{-1})$ respectively. Then this investor sends bullish, bearish or neutral signal to his nearest neighbors. According to one-dimensional voter process system, investors can affect each other or the news can be spread during the daily trading time $[0, T]$, which is assumed as the main factor of price fluctuations for type-2 investors. For a fixed $s \in [0, T]$ (large enough), let

$$B(\omega_k) = \text{sgn}_k(0)|\xi^{[0]}_{s,t}||m_2$$

where $|\xi^{[0]}_{s,t}|$ is the cardinality of $\xi^{[0]}_{s,t}$, which is expressed by

$$|\xi^{[0]}_{s,t}| = \sum_{y=-m(t)/2}^{m(t)/2} |\xi^{[0]}_{s,t}(y)|.$$  

From above definitions, we define the stock price of type-2 at time $n(t = 1, 2, \cdots, n)$ as

$$S_{i}^{type-2}(s) = e^{\alpha S_{i}^{type-2}(s)}S_{i-1}^{type-2}(s)$$

where $\alpha > 0$. Then we have

$$S_{i}^{type-2}(s) = S_0 \exp[\alpha \sum_{k=1}^{n(t)} B(\omega_k)].$$

$$S_i = S_0 \exp[\alpha \sum_{k=1}^{n(t)} A(\omega_k)] + \frac{1}{\sqrt{n}} \alpha \sum_{k=1}^{n(t)} B(\omega_k)].$$

$$\text{III. THE LIMITING PROPERTIES OF STOCK PRICES DISTRIBUTION}$$

In Section 2 of the present paper, we give the model of stock prices in the form of (9), now we continue to discuss the process with the continuous time. So in that case, the normalized stock process $Y^n_v, v \in [0,1]$ is defined by

$$Y^n_v = \frac{1}{\sqrt{n}} \sum_{k=1}^{n(t)} A(\omega_k), v \in [0,1]$$

where $\lfloor sv \rfloor$ denote the integral part of a real number $nv$, and $Q_v$ is a random variable which denote the area under the stock price. More specifically, for type-1 investors, considering the random walk $\{X_t = \alpha \sum_{k=1}^{n(t)} A(\omega_k), t = 0, 1, \cdots\}$ (see [4,17,18]), and for a fixed $n$, let

$$Q_v = \frac{1}{n} \sum_{k=1}^{n(t)} A(\omega_k) = \alpha \sum_{k=1}^{n(t)} (1 - \frac{1}{n}) A(\omega_k).$$

be a new random variable which denote the area under a random polygonal function $l_u(v), v \in [0,1]$ defined by

$$l_u(v) = X_{\lfloor nv \rfloor} + \{nv\} A(\omega_{\lfloor nv \rfloor+1}), v \in [0,1]$$

where $\lfloor nv \rfloor$ denote the fractional part of $nv$. From (2), we know that the stock price $S_{i}^{type-2} = S_0 \exp[\alpha \sum_{k=1}^{n(t)} A(\omega_k)]$ is determined by the random polygonal line.

In a stock market, here the ‘area’ $Q_v$ may represent the situation on this stock during the time from 0 to $n$, including the investors’ estimation for the price of this stock, the positive or negative news, trends, political event and economic policy, etc. For example, if the ‘area’ is positive, there may have a positive influence on some market participants so that they are likely to take buying positions. In this paper, we only consider the case that the ‘area’ is positive, and we can use the similar methods to discuss the opposite case. In the following, on some condition (see [4,17,18]), the limiting probability distribution of the random process $\{\alpha (\sum_{k=1}^{n(t)} A(\omega_k), t = 0, 1, \cdots\}$ is given. The following results are based on the work of [1,3,4,10,17,18], that is, when $\beta$ is large enough, the finite dimensional distribution of the conditional random process $\{\alpha (\sum_{k=1}^{n(t)} A(\omega_k), t = 0, 1, \cdots\}$ converges to the corresponding distribution of random process

$$\{\int_0^t \mu(u) \mathrm{d}u + \int_0^t \sigma(\beta) \mathrm{dB}(u)\}$$

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where $B(u)$ is a standard Brownian motion, $\sigma_t(\beta)$ is the degree of fluctuation of the stock price, and $\mu(u)$ is the local tendency of the stock price process. The convergence of (11) has been shown in [17], and $\sigma_t(\beta)$, $\mu(u)$ are given in [17].

Next we consider the convergence of $[\alpha \sum_{i=1}^{m} B(\omega_i)]/\sqrt{n, \nu} \in [0,1]$, the second part in (10). Suppose that $B(\omega_i) = \text{sgn}(0)|z|/m_z(t)$, where $m_z(t)$ depends on the trading days $n$. Then according to the theory of voter model (see [7]), we have

(a) If $\lambda < \lambda_0$, there is a $\rho > 0$ such that

$$P(\xi^{(0)}(s) \neq \emptyset) \leq e^{-\rho s}$$

(12)

then the process dies out exponentially fast.

(b) If $\lambda > \lambda_0$, then on $\{\xi^{(0)}(s) \neq \emptyset, \text{ for all } s \geq 0\}$,

$$\frac{|\xi^{(0)}(s)|}{s} \rightarrow 2(\lambda-1), \text{ a.s., as } s \rightarrow \infty.$$  

(13)

(c) For any $\epsilon > 0$ and $m_z(n)$ large enough, if $\lambda < \lambda_0$,

$$E\left(\frac{|\xi^{(0)}(s)|}{m_z(n)}\right) < \epsilon, \text{ as } s \rightarrow \infty.$$  

(14)

(d) For $m_z(n)$ large enough, if $\lambda > \lambda_0$, there is a $\rho > 0$ such that, as $s \rightarrow \infty$,

$$E\left(\frac{|\xi^{(0)}(s)|}{m_z(n)}\right) \geq \rho, \quad E\left(\frac{|\xi^{(0)}(s)|}{m_z(n)}\right)^2 \geq \rho.$$  

(15)

Next we define the stopping time for type-2 investors in the prices model, this implies that the type-2 investors also decide their trading positions by the historical information. This shows that the price model (9)(10) is a non-Markovian process. Let $\tau_1, \tau_2, \ldots, \tau_n, \ldots$, denote the stopping defined as followings

$$\tau_1 = \min\{k \geq 1; \; n^{-1/2} \sum_{i=1}^{k} B(\sigma_i) \geq 1\}, \ldots$$

$$\tau_n = \min\{k \geq 1; \; n^{-1} \sum_{i=1}^{\tau_{n-1}} B(\sigma_i) \geq 1\}, \ldots$$

(16)

For every stopping time intervals $[\tau_{n-1} + 1, \tau_n]$, define a $\lambda_n > 0$ on this time interval, such that for some $0 < \alpha < 1$, if $m \leq n^\alpha / 2$ then $\lambda_n < \lambda_0$, if $m > n^\alpha / 2$ then $\lambda_n > \lambda_0$. Then we have the following results. For any fixed $k$, 

$$E[\mathcal{B}(\sigma_i)] = (p_k - q_k)E\left(\frac{|\xi^{(0)}(s)|}{m_z(k)}\right)$$

(17)

$$|E[\mathcal{B}(\sigma_i)]|^2 = (p_k + q_k)E\left(\frac{|\xi^{(0)}(s)|}{m_z(k)}\right)^2$$

(18)

By (12)-(18), if $\lambda < \lambda_0$, we can properly choose $p_k, q_k$, such that $E[\mathcal{B}(\sigma_i)] = 0$, such as $n \rightarrow \infty$.

If $\lambda > \lambda_0$, then $\mathcal{B}(\sigma_i)$ diverges as $n \rightarrow \infty$. In order to show the convergence of the distribution, we consider the convergence of the characteristic function of $\mathcal{B}_n$, i.e., 

$$\varphi_n(z) = \mathcal{E}\left[\exp\{iz\mathcal{B}_n\}\right], \text{ as } n \rightarrow \infty$$

(19)

where $c$ is a positive constant.

Taking the scaling limit of the second part in the discrete time model (9), we will obtain a continuous time process, and we discuss the probability distribution of this continuous time model. Let

$$0 < \nu < 1, \text{ } [n^\nu] \in [1 + \tau_1 + \ldots + \tau_{n-1}, \tau_1 + \ldots + \tau_n]$$

(20)

where $[n^\nu]$ is the integer part of $n^\nu$. Then $m$ can be expressed by $m = m(n, v)$, let

$$B^*_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{k-\nu \tau_{n-1}} B(\sigma_i), 0 < \nu < 1.$$  

(21)

Now we define the prices model I terms of above (21) by

$$G(n, v) = G(0) \exp\{\mathcal{B}_n^*\}, \text{ } 0 < \nu < 1.$$  

(22)

where $G(0)$ is an initial state at time 0.

Then we have, as $n \rightarrow \infty$, the probability distribution of the process $G(n, v)$ convergence to the corresponding distribution of

$$G(0) \exp\{\int_0^t \mu(u)\mathcal{B}(u)du + \int_0^t \sigma(\mathcal{B}(u)du\}, 0 < \nu < 1$$

(23)

where $B(u)$ is the one dimensional standard Brownian motion, $\mu(u)$ is the trend function and $\sigma(u)$ is the volatility function. The limit (23) will be shown in the next section.

IV. THE EXISTENCE OF THE ASYMPTOTICAL BEHAVIOR OF PRICE MODEL FOR THE TYPE-2 INVESTORS

In this section, we will show that the normalized price model $G(n, v)$ convergence to the corresponding distribution of (23). In order to show the convergence of the distribution, we consider the convergence of the characteristic function of $\mathcal{B}_n^*$, i.e., 

$$\varphi_n^*(z) = \mathcal{E}\left[\exp\{iz\mathcal{B}_n^*\}\right], \text{ as } n \rightarrow \infty$$

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where \( i = \sqrt{-1} \). \( \phi(z) \) is divided into two terms as follows
\[
\phi(z) = E[\exp\{izB_n^{\alpha}\}; \tau_n - n^{5/6} \leq n^{2/3+\epsilon},
\]
for all \( m = 1, \cdots, n^n/2 \)
\[+ E[\exp\{izB_n^{\alpha}\}; \tau_n - n^{5/6} > n^{2/3+\epsilon}, \]
for some \( m = 1, \cdots, n^n/2 \). (24)

Next we define the conditional expectations,
\[
K_1 = E[\exp\{izB_n^{\alpha}\}; \tau_n - n^{5/6} \leq n^{2/3+\epsilon},
\]
for all \( m = 1, \cdots, n^n/2 \) (25)
\[
K_2 = E[\exp\{izB_n^{\alpha}\}; \tau_n - n^{5/6} > n^{2/3+\epsilon},
\]
for some \( m = 1, \cdots, n^n/2 \). (26)

(I) Now we estimate the second term \( K_2 \). Let \( 1/12 < \varepsilon < 1/6 \) and \( B_k = \sum_{k=1}^n B(\sigma_i) \), then
\[
P(\tau_n - n^{5/6} > n^{2/3+\epsilon})
= P(\tau_n > n^{2/3+\epsilon} + n^{5/6}) + P(\tau_n < n^{5/6} - n^{2/3+\epsilon})
= P(B_{n^{1/3+\epsilon}} \leq n^{1/3}) + P(B_{n^{1/3+\epsilon}} \geq n^{1/3})
= P((B_{n^{1/3+\epsilon}} - E[B_{n^{1/3+\epsilon}}]) \leq -n^{1/6+\epsilon})
+ P((B_{n^{1/3+\epsilon}} - E[B_{n^{1/3+\epsilon}}]) \geq n^{1/6+\epsilon})
\leq (n^{1/3} - n^{1/6+\epsilon})/n^{1/3+2\epsilon} + (n^{1/3} + n^{1/6+\epsilon})/n^{1/3+2\epsilon}
= \frac{2}{n^{\epsilon}}. 
\] (27)

By (26) and (27), when \( \alpha = 1/6 \), then we have \( K_2 \leq n^{5/6}/n^{\epsilon} \), so that
\[
K_2 \to 0, \quad \text{as } n \to \infty .
\]
This implies that the length of stopping time \( \tau_n \) is about \( n^{5/6} \).

(II) From above discussion, for \( \alpha = 1/6 \), \( n^n/2) \times n^{5/6} = n/2 \). So, when \( k \leq n/2 \), \( \lambda < \lambda \); when \( k > n/2 \), \( \lambda > \lambda \).

(a) If \( m \leq n^n/2 \), and \( k \leq n/2 \), by (19) we have
\[
E[\exp\{iz\sqrt{n}/\sqrt{n}B(\sigma_i)\}]
= 1 + iz\sqrt{n}/\sqrt{n} E[B(\sigma_i)] - \frac{z^2}{2n} E[B(\sigma_i)]^2 + o(\frac{1}{n^{1/2}})
= 1 + iz\sqrt{n}/\sqrt{n} + o(\frac{1}{n^{1/2}}).
\] (28)

(b) If \( m > n^n/2 \), and \( k > n/2 \), by (20) we have
\[
E[\exp\{iz\sqrt{n}/\sqrt{n}B(\sigma_i)\}]
= 1 + iz\sqrt{n}/\sqrt{n} E[B(\sigma_i)] - \frac{z^2}{2n} E[B(\sigma_i)]^2 + o(\frac{1}{n^{1/2}})
= 1 + iz - \frac{z^2}{2n} + o(\frac{1}{n^{1/2}}).
\] (29)

(III) We estimate the first term \( K_1 \), in two parts:

(a) If \( 0 < \varepsilon < 1/2 \), \( m(n,v) = [n^{1/6}] \), by (28) we have
\[
K_1 = \sum_{m=1}^{[n^{1/6}]} E[\exp\{iz\sqrt{n}/\sqrt{n}B(\sigma_i)\}; \tau_n = \tau_m, \quad \text{for all } m = 1, \cdots, n^n/2]
\]
\[
= \sum_{m=1}^{[n^{1/6}]} \prod_{m < n} E[\exp\{iz\sqrt{n}/\sqrt{n}B(\sigma_i)\}] \cdot
\]
\[
= \sum_{m=1}^{[n^{1/6}]} \prod_{m < n} (1 + iz\sqrt{n}/\sqrt{n} + o(\frac{1}{n^{1/2}})) \cdot
\]
\[
= \sum_{m=1}^{[n^{1/6}]} \exp[\sum_{m=1}^{[n^{1/6}]} r_m \ln(1 + iz\sqrt{n}/\sqrt{n} + o(\frac{1}{n^{1/2}}))]
\]
\[
= \sum_{m=1}^{[n^{1/6}]} \exp[iz\sqrt{n}/\sqrt{n} + o(\frac{1}{n^{1/2}})].
\] (30)

On the other hand, by (27) we have
\[
P(\tau_n - n^{5/6} \leq n^{2/3+\epsilon}, m = 1, \cdots, n^n/2)
= \prod_{m=1}^{[n^{1/6}]} P(\tau_n - n^{5/6} \leq n^{2/3+\epsilon})
\]
\[\prod_{n=1}^{[n^2/2]} (1 - n^{-2\varepsilon}) = (1 - n^{-2\varepsilon})^{n^2/2}. \]  
(31)

So for \( \alpha = 1/6 \) and \( 1/12 < \varepsilon < 1/6 \), then \( 2\varepsilon > \alpha \), so that, as \( n \to \infty \),

\[\ln(1 - n^{-2\varepsilon})^{n^2/2} \approx \frac{1}{2} \frac{n^\alpha}{n^{2\varepsilon}} = \frac{n^\alpha}{2n^{2\varepsilon}} \to 0.\]

Then we have

\[\lim_{n \to \infty} P(|\tau_n - n^{5/6}| \leq n^{2/3+\varepsilon}, m = 1, \ldots, \frac{n^\alpha}{2}) = 1. \]  
(32)

Combining (24)-(26) and (30)-(32), if \( 0 < \varepsilon < 1/2 \), then we have

\[\lim_{n \to \infty} \phi^*_n(z) = \lim_{n \to \infty} \exp[i\varepsilon + o(\frac{1}{n^{1-\varepsilon}})] = \exp[i\varepsilon]. \]  
(33)

(b) If \( 1/2 \leq \varepsilon < 1 \) \((k > n/2)\), following the similar procedure of above (a), and by (28)(29), we have

\[E[\exp[i\varepsilon B_n^*]] | \tau_n = \tau_m, \text{ for all } m = 1, \ldots, \frac{n^\alpha}{2}\]

\[= \prod_{m=1}^{[n^2/2]} E[\exp(i\varepsilon B_\sigma)] | (u < n/2) \times E[\exp(i\varepsilon B_\sigma)]^{m-1 + \varepsilon \sigma} \]

\[= (1 + i\varepsilon + o(\frac{1}{\sqrt{n}}))^{(m-1 + \varepsilon \sigma)} \times (1 + i\varepsilon - \frac{z^2c}{2} + o(\frac{1}{\sqrt{n}}))^{(m-1 + \varepsilon \sigma)} \]

\[= \exp[i\varepsilon - \frac{1}{2}(v-\frac{1}{2})z^2c + o(\frac{1}{\sqrt{n}})]. \]

Then we have, for \( 1/2 \leq \varepsilon < 1 \)

\[\lim_{n \to \infty} \phi^*_n(z) = \lim_{n \to \infty} \exp[i\varepsilon - \frac{1}{2}(v-\frac{1}{2})z^2c + o(\frac{1}{\sqrt{n}})] = \exp[i\varepsilon - \frac{1}{2}(v-\frac{1}{2})z^2c]. \]  
(34)

(IV) Combining (33) and (34), for \( 0 < \varepsilon < 1 \), we have

\[\lim_{n \to \infty} \phi^*_n(z) = \lim_{n \to \infty} E[\exp[i\varepsilon B_n^*]] = \exp[i\varepsilon \mu_2(v) - \frac{1}{2} \sigma_2^2(v)(v-\frac{1}{2})z^2]. \]  
(35)

where \( \mu_2(v) = 1 \), and \( \sigma_2^2(v) = 0 \) if \( 0 < v < 1/2 \), \( \sigma_2^2(v) = c \) if \( 1/2 \leq v < 1 \).

By [1], above (35) shows that the probability distributions of prices model of the present paper converge to the corresponding distributions of a geometric Brownian motion, this completes the proof of (23).

V. THE ASYMPTOTICAL DISTRIBUTION OF STOCK PRICE MODEL

In Section 3 and Section 4, we discussed the convergence of the normalized prices process \( Y_n, n \in [0,1] \) (see (10)). In conclusion, by (10)(11)(23), and according to the probability distribution convergence theory, see [1,20], we have, for \( v \in [0,1] \) and \( n \to +\infty \), the probability distribution of the normalized stock price \( S_n \exp(Y_n) \) defined in (10) converges to the corresponding probability distribution of random process

\[S_n = S_0 \exp \left[ \int_0^T \left( \mu(s) + \frac{1}{2} \sigma^2(s) \right) ds + \int_0^T \sigma(s) dB(s) \right]. \]  
(36)

Above (36) is usually called the Black-Scholes model. The Black-Scholes model is a well known model in finance, it is a continuous time model with one risky asset and a riskless asset, see [6,8,11]. The model was suggested by Black and Scholes to describe the behavior of prices. Black and Scholes were the first to suggest a model whereby we can derive an explicit price for a European call on a stock that pays no dividend. According to their model, the writer of the option can hedge himself perfectly, and actually the call premium is the amount of money needed at time 0 to replicate exactly the payoff \((S_T - K)_+\), by following their dynamic hedging strategy until maturity, where \( T \) is the expiration date and \( K \) is the exercise price. Moreover, the formula depends on only one non-directly observable parameter, the so-called “volatility”.

VI. VALUATION AND HEDGING OF EUROPEAN CONTINGENT CLAIMS

In this section, we discuss the corresponding valuation and hedging of European contingent claims for the prices model given in Section 2-3. For the options, they have been the main motivation in the construction of the theory and still are the most spectacular example of the relevance of applying stochastic theory to finance. An option gives its holder the right, but not the obligation, to buy or sell a certain amount of a financial asset, by a certain date, for a certain strike price. The writer of the option needs to specify: (a) the type of option, the option to buy is called a call while the option to sell is a put; (b) the underlying asset, typically, it can be a stock, a bond, a currency and so on; (c) the amount of an underlying asset to be purchased or sold; (d) the expiration date, if the option can be exercised at any time before maturity, it is called an American option but, if it can only be exercised
at maturity, it is called a European option; (e) the exercised price which is the price at which the transaction is done if the option is exercised.

The price of the option is the premium. When the option is traded on an organized market, the premium is quoted by the market. Otherwise, the problem is to price the option. Also, even if the option is traded on an organized market, it can be interesting to detect some possible abnormalities in the market. In this paper, the results concentrates on European call option on a stock, whose price at time \( t \) is denoted \( S_t \).

And it is known that, for example see [6,8,11], if (36) is a risk-neutral geometric Brownian motion, then the drift function of (36) must satisfy the following condition

\[
\mu_t(u) + \mu_s(u) = r - \frac{1}{2}\left(\sigma_t(\beta) + \sigma_s(u)\right)^2
\]

(37)

where \( r \) is the continuously compounded nominal interest rate, a riskless parameter. In the construction of our financial model in Section 2-4, we don't consider the risk-neutral probability distribution, so the limiting stock process defined in (10) is not a risk-neutral process unless we have the condition (37). So that, this makes all security-buying and security-selling bets fair, and its valuation of cost of the option can be precisely as given by the Black-Scholes formula.

One of the main features of the Black-Scholes model (and one of the reasons for its success) is the fact that the pricing formulae, as well as the hedging formulae see [2,8,11,12,15,19], depend on only one non-observable parameter, called “volatility” by practitioners (the drift parameter disappears by change of probability). According to the theorem of representation of Brownian martingales [8, 15, 19], it can show the existence of a replicating portfolio. When the option is defined by a random variable \( h = \nu(S_t) \), we show that it is possible to find an explicit hedging portfolio for the European option in the following.

Now we discuss the pricing and hedging of contingent claim on the Black-Scholes model given in (36), and we focus on the European contingent claim \( \nu(S_t) \). Let \( V(\nu, S_t) \) denote the value of the European contingent claim with inception time \( \nu \) and expiration time \( T \), then according to the theory of Black-Scholes valuation and the Girsanov theorem (see [6,8,11]), the value \( V(\nu, S_t) \) of the portfolio at time \( \nu \) corresponding to the contingent claim \( V(T, S_t) = \nu(S_t) \) is given by

\[
V(\nu, S_t) = e^{-r(\nu - t)} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \nu(S_t e^{(T-t)\beta}) e^{-\frac{(\sigma^2\nu^2)(T-t)}{2}} \exp\left(-\frac{\nu^2}{2}\right) d\nu
\]

(38)

where \( \sigma = \sigma_t(\beta) + \sigma_s(\nu) \) is given in (36). The equation (21) can be obtained following the argument in [6,8,11], here we omit the derivation. Since \( \sigma \) depends on the parameters \( \beta, \gamma, \delta, d, \lambda \), the value of \( V(\nu, S_t) \) also depends on these parameters. This implies that the value \( V(\nu, S_t) \) depends on the activity of the stock market, investors' trading positions and trading strategies, the market sentiment and the speed of spread news in this financial model. Further, the value \( V(\nu, S_t) \) is determined by the professional investors and nonprofessional investors in the financial model of the present paper.

VII. CONCLUSION

In the present paper, the stochastic stock price is modeled by the interacting particle dynamic systems in Section 1 and Section 2. In the model, the investors of stock markets are divided into two types of investors, type-1 investors (the professional investors), type-2 investors (the general investors or nonprofessional investors). We think that this kind of research is a new approach to study the statistical properties of fluctuations of stock market. In order to make the financial model to describe the stock market more properly, we introduce the stopping time theory in Section 3 to study the fluctuations of stock price. In Section 4 and Section 5, we show the convergence of probability distributions of the normalized prices process, where the stochastic process theory is applied to show the results of this paper. Further in Section 6, we study the option pricing under this financial model, and give the corresponding valuation and hedging of European contingent claims. We hope that this research work is helpful for us to understand the statistical properties of fluctuations of stock price in globalized securities markets, and useful for valuation and hedging of European call option. The research work of the present paper is mainly base on the theory of the interacting particle systems, Gibbs measure and voter model. Note that this kind of work is a new approach or a new method to study the statistical behavior of stock market.

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