Efficient Numerical Computations of Soft Constrained Nash Strategy for Weakly Coupled Large-Scale Systems

Muneomi Sagara, Hiroaki Mukaidani, Toru Yamamoto
Graduate School of Education, Hiroshima University, Higashihiroshima, JAPAN
Email: {d074978, mukaida, yama}@hiroshima-u.ac.jp

***Abstract*— In this paper, a high-order soft constrained Nash strategy for weakly coupled large-scale systems is investigated. In order to solve the cross-coupled sign-indefinite algebraic Riccati equations (CSAREs) corresponding to strategy, the iterative algorithm on the basis of the Newton's method is first applied. Second, the recursive algorithm for solving the CSAREs is also established to reduce the amount of algebraic computation as compared with the Newton's method. Using these iterative solutions, a high-order soft-constrained Nash strategy is designed. As a result, it is proved that the proposed high-order approximate equilibrium strategies achieve better performance. Finally, in order to demonstrate the efficiency of the algorithm, a numerical example is given.**

***Index Terms*—soft constrained Nash strategy, weakly coupled large-scale systems, cross-coupled sign-indefinite algebraic Riccati equations (CSAREs), Newton's method, recursive algorithm, high-order approximate equilibrium strategies**

I. Introduction

The control problems of weakly coupled large-scale systems have been studied by several researchers (see [3], [4] and references therein). In practice, it is known that such systems are represented by multi-area power systems [3], distillation columns [11] and cold-rolling mills [12]. They are widely used to represent the system dynamics.

Linear quadratic Nash games and their applications have been widely investigated in many literatures (see e.g. [5], [6] for weakly coupled large-scale systems). In the deterministic case, a guaranteed cost problem for the state feedback strategies of nonzero-sum differential games involving multiple players was investigated [13]. The Nash strategy was applied to systems using active magnetic bearings: it has successfully reduced the error dynamics induced by linearization [14]. However, robust Nash equilibrium in deterministic uncertain systems has not been investigated thus far. In contrast, robust equilibria in indefinite linear quadratic differential games under a deterministic disturbance input affecting the systems have been discussed [1]. The results in [1] are very elegant in theory and it is easy to obtain a strategy pair by solving the cross-coupled sign-indefinite algebraic Riccati equations (CSAREs). In [2], the numerical algorithm that is based on the calculation of the eigenstructure for solving the soft-constrained Nash equilibria has been developed. However, the scalar case has only been considered. In addition, the convergence rate is unclear. On the other hand, the Newton-type algorithm for solving the CSAREs has been established [7], [8]. Although the Newton's method guarantees the fast convergence, this existing algorithm has to utilize two fixed-point iterations to attain the reduced-order computations. Therefore, the large amount of computation and CPU time are needed. Moreover, the degradation of the cost performance under the use of the strategies that is based on the iterative solution has not been investigated in [7].

This paper investigates the soft-constrained Nash games for the weakly coupled large-scale systems that attains the high-order approximation by using the iterative solutions. The main contribution is to propose a new high-order approximation strategy that is based on the solution of the CSAREs via the Newton's method with the appropriate initial guess. As a result, although the CSAREs has the sign-indefinite quadratic form as compared with the existing results [5], [6], the convergence solutions satisfy the required properties such as the stability and the positive-definiteness. Thus, it is proved that the proposed strategy that is based on the iterative solutions with a few iterations achieves a high-order approximation of better equilibrium. As another important contribution, the recursive algorithm for solving the CSAREs is given. It is shown that the proposed algorithm converges to the exact solution with linear rate. In addition, since only low-order systems are involved in algebraic computations, the amount of computations required does not increase per iterations as compared with the Newton's method [7], [8]. As a result, the reduction of the CPU time is guaranteed. Finally, in order to demonstrate the efficiency of the proposed design methodology, a numerical example is included.

Notation: The notations used in this paper are fairly standard. The superscript T denotes the matrix transpose. I_n denotes the $n \times n$ identity matrix. $\mathbf{block\ diag}$ denotes the block diagonal matrix. $\| \cdot \|$ denotes its Euclidean norm for a matrix. $\otimes$ denotes the Kronecker product. δ_{ij} denotes the Kronecker delta. $\mathrm{vec} M$ denotes the column

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vector of the matrix M. The space of $\mathbf{R}^k$-valued functions that are quadratically integrable on $(0,\ \infty)$ is denoted by $L_2^k(0,\ \infty)$.

II. Problem Formulation

Consider the weakly coupled large-scale linear systems with N-players

$$\dot{x}_i(t) = A_{ii}x_i(t) + B_{ii}u_i(t) + \varepsilon \sum_{j=1,\ j\neq i}^{N} A_{ij}x_j(t) + \varepsilon \sum_{j=1,\ j\neq i}^{N} B_{ij}u_j(t) + E_{ii}w_i(t) + \varepsilon \sum_{j=1,\ j\neq i}^{N} E_{ij}w_j(t),$$
$$x_i(0) = x_i^0,\quad i = 1,\ \dots,N, \tag{1}$$

where $x_i \in \mathbf{R}^{n_i},\ i = 1,\ \dots,N$ represent i-th state vectors. $u_i \in \mathbf{R}^{m_i},\ i = 1,\ \dots,N$ represent i-th control inputs. $w_i \in \mathbf{R}^{k_i},\ i = 1,\ \dots,N$ represent i-th disturbance vectors. ε denotes a small positive weak coupling parameter which connect the other subsystems.

Let us introduce the partitioned matrices

$$A_\varepsilon := \begin{bmatrix} A_{11} & \varepsilon A_{12} & \cdots & \varepsilon A_{1N} \\ \varepsilon A_{21} & A_{22} & \cdots & \varepsilon A_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon A_{N1} & \varepsilon A_{N2} & \cdots & A_{NN} \end{bmatrix},$$

$$B_{i\varepsilon} := \begin{bmatrix} \varepsilon^{1-\delta_{1i}}B_{1i} \\ \varepsilon^{1-\delta_{2i}}B_{2i} \\ \vdots \\ \varepsilon^{1-\delta_{Ni}}B_{Ni} \end{bmatrix},\ \delta_{ij} := \begin{cases} 0\ (i \neq j) \\ 1\ (i = j) \end{cases},$$

$$E_\varepsilon := \begin{bmatrix} E_{11} & \varepsilon E_{12} & \cdots & \varepsilon E_{1N} \\ \varepsilon E_{21} & E_{22} & \cdots & \varepsilon E_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon E_{N1} & \varepsilon E_{N2} & \cdots & E_{NN} \end{bmatrix}.$$

By using above relations, the system (1) can be changed as

$$\dot{x}(t) = A_\varepsilon x(t) + \sum_{i=1}^{N} B_{i\varepsilon}u_i(t) + E_\varepsilon w(t), \tag{2}$$

where

$$x(t) := \begin{bmatrix} x_1(t)^T & \cdots & x_N(t)^T \end{bmatrix}^T \in \mathbf{R}^{\bar{n}},\ \bar{n} := \sum_{i=1}^{N} n_i,$$
$$w(t) := \begin{bmatrix} w_1(t)^T & \cdots & w_N(t)^T \end{bmatrix}^T \in \mathbf{R}^{\bar{k}},\ \bar{k} := \sum_{i=1}^{N} k_i.$$

The cost performance for each strategy subset is defined by

$$J_i(u_1,\ \dots,u_N,\ w,\ x(0)) = \int_0^\infty \Big[x^T(t)Q_{i\varepsilon}x(t) + u_i^T(t)R_{ii}u_i(t) + \mu \sum_{j=1,\ j\neq i}^{N} u_j^T(t)R_{ij}u_j(t) - w^T(t)V_{i\mu}w(t)\Big]dt, \tag{3}$$

where

$$Q_{i\varepsilon} = \begin{bmatrix} \varepsilon^{1-\delta_{i1}}Q_{i1} & \varepsilon Q_{i12} & \cdots & \varepsilon Q_{i1N} \\ \varepsilon Q_{i12}^T & \varepsilon^{1-\delta_{i2}}Q_{i2} & \cdots & \varepsilon Q_{i2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon Q_{i1N}^T & \varepsilon Q_{i2N}^T & \cdots & \varepsilon^{1-\delta_{iN}}Q_{iN} \end{bmatrix} \in \mathbf{R}^{\bar{n}\times\bar{n}},$$
$$R_{ii} = R_{ii}^T > 0 \in \mathbf{R}^{m_i\times m_i},\ R_{ij} = R_{ij}^T \geq 0 \in \mathbf{R}^{m_j\times m_j},$$
$$V_{i\mu} = \textbf{block diag}\begin{pmatrix} \mu^{-(1-\delta_{i1})}V_{i1} & \cdots & \mu^{-(1-\delta_{iN})}V_{iN} \end{pmatrix} \geq 0 \in \mathbf{R}^{\bar{k}\times\bar{k}},\ i,\ j = 1,\ \dots,N.$$

The state weight matrices $Q_{i\varepsilon}$ is symmetric and assumed to be sign-indefinite [2]. Furthermore, it should be noted that μ denotes a small positive parameter which is the same order for the parameter ε. That is, the following assumption is made.

Assumption 1: The ratio of the small positive parameters ε and μ is bounded by some positive constants $\tilde{k}$.

$$0 < \tilde{k} := \frac{\mu}{\varepsilon} < \infty. \tag{4}$$

Although it seems that this assumption is conservative, it will not affect the main contribution from the practical points of view.

A. Soft Constrained Nash Equilibrium Strategy

For the matrices $A_\varepsilon,\ B_{i\varepsilon},\ i = 1,\ \dots,\ N$, the set $\boldsymbol{F}_N$ is defined by

$$\boldsymbol{F}_N := \left\{(F_{1\varepsilon},\ \dots,F_{N\varepsilon})\ \middle|\ A_\varepsilon + \sum_{j=1}^{N} B_{j\varepsilon}F_{j\varepsilon}\ \text{is stable.}\right\}.$$

The soft constrained Nash equilibrium strategy pair $(F_{1\varepsilon}^*,\ \dots,F_{N\varepsilon}^*)$ is defined as satisfying the following conditions [2].

$$\bar{J}_i(u_1^*,\ \dots,u_N^*,\ x(0)) \leq \bar{J}_i(u_1^*,\ \dots,\ u_i,\ \dots,\ u_N^*,\ x(0)),\ i = 1,\ \dots,N, \tag{5}$$

where $u_i := u_i(t) = F_{i\varepsilon}x(t)$,

$$\bar{J}_i(F_{1\varepsilon}x,\ \dots,\ F_{N\varepsilon}x,\ x(0)) := \sup_{w\in L_2^{\bar{k}}(0,\ \infty)} J_i(F_{1\varepsilon}x,\ \dots,\ F_{N\varepsilon}x,\ w,\ x(0)),$$
$$J_i(F_{1\varepsilon}x,\ \dots,F_{N\varepsilon}x,\ w,\ x(0)) = \int_0^\infty \Big[x^T(t)[Q_{i\varepsilon} + F_{i\varepsilon}^T R_{ii}F_{i\varepsilon} + \mu \sum_{j=1,\ j\neq i}^{N} F_{j\varepsilon}^T R_{ij}F_{j\varepsilon}]x(t) - w^T(t)V_{i\mu}w(t)\Big]dt,$$

for all $x(0)$ and for all $(F_{1\varepsilon},\ \dots,F_{N\varepsilon})$ that satisfy

$$(F_{1\varepsilon}^*,\ \dots,\ F_{i-1\varepsilon}^*,\ F_{i\varepsilon},\ F_{i+1\varepsilon}^*,\ \dots,\ F_{N\varepsilon}^*) \in \boldsymbol{F}_N.$$

It should be noted that the following assumption guarantees the existence of the admissible strategy.

Assumption 2: Each player uses the linear feedback strategy $u_i(t) = K_{i\varepsilon}x(t),\ i = 1,\ \dots,N$ such that the closed-loop system is asymptotically stable for sufficiently small parameters ε and μ.

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This assumption is imposed on each player restrictively. In other words, it is assumed that each player on the subsystem take a stable strategy. If such conditions are not met, the overall systems would be not asymptotically stable.

B. CSAREs

Using the fact studied by [2], the soft constrained feedback Nash equilibrium is given below.

Lemma 1: [2] Assume that there exist N real symmetric matrices $P_{i\varepsilon}$ and $W_{i\varepsilon}$, such that

$$\begin{aligned}&\boldsymbol{G}_i(P_{1\varepsilon},\ \dots,P_{N\varepsilon})\\ &:= P_{i\varepsilon}\left(A_\varepsilon - \sum_{j=1}^{N} S_{j\varepsilon}P_{j\varepsilon}\right) + \left(A_\varepsilon - \sum_{j=1}^{N} S_{j\varepsilon}P_{j\varepsilon}\right)^T P_{i\varepsilon}\\ &\quad + P_{i\varepsilon}S_{i\varepsilon}P_{i\varepsilon}\\ &\quad + \mu \sum_{j=1,\ j\neq i}^{N} P_{j\varepsilon}S_{ij\varepsilon}P_{j\varepsilon} + P_{i\varepsilon}M_{i\mu}P_{i\varepsilon} + Q_{i\varepsilon} = 0, \quad (6)\end{aligned}$$

where $S_{i\varepsilon} := B_{i\varepsilon}R_{ii}^{-1}B_{i\varepsilon}^T$, $S_{ij\varepsilon} := B_{j\varepsilon}R_{jj}^{-1}R_{ij}R_{jj}^{-1}B_{j\varepsilon}^T$, $M_{i\mu} := E_\varepsilon V_{i\mu}^{-1}E_\varepsilon^T$.

$A_\varepsilon - \sum_{j=1}^{N} S_{j\varepsilon}P_{j\varepsilon} + M_{i\mu}P_{i\varepsilon}$ is stable for $i = 1,\ \dots,N$,

$A_\varepsilon - \sum_{j=1}^{N} S_{j\varepsilon}P_{j\varepsilon}$ is stable,

$$\begin{aligned}&W_{i\varepsilon}\left(A_\varepsilon - \sum_{j=1,\ j\neq i}^{N} S_{j\varepsilon}P_{j\varepsilon}\right) + \left(A_\varepsilon - \sum_{j=1,\ j\neq i}^{N} S_{j\varepsilon}P_{j\varepsilon}\right)^T W_{i\varepsilon}\\ &\quad - W_{i\varepsilon}S_{i\varepsilon}W_{i\varepsilon} + \mu \sum_{j=1,\ j\neq i}^{N} P_{j\varepsilon}S_{ij\varepsilon}P_{j\varepsilon} + Q_{i\varepsilon} \geq 0. \quad (7)\end{aligned}$$

Define the N-tuple $(F_{1\varepsilon}^*,\ \dots,F_{N\varepsilon}^*)$ by

$$u_i^*(t) := F_{i\varepsilon}^*x(t) = -R_{ii}^{-1}B_{i\varepsilon}^TP_{i\varepsilon}x(t),\ i = 1,\ \dots,N. \quad (8)$$

Then, $(F_{1\varepsilon}^*,\ \dots,F_{N\varepsilon}^*) \in \boldsymbol{F}_N$ and this N-tuple is a soft constrained Nash equilibrium. Furthermore, $\bar{J}_i(F_{1\varepsilon}^*x,\ \dots,\ F_{N\varepsilon}^*x,\ x(0)) = x(0)^TP_{i\varepsilon}x(0)$.

It should be noted that if $Q_{i\varepsilon} \geq 0$ and $S_{ij\varepsilon} \geq 0$ for all $i = 1,\ \dots,N$, the matrix inequality (7) is trivially satisfied with $W_{i\varepsilon} = 0$ [2]. Then, only the CSAREs (6) should be solved.

In the following analysis, the basic assumption is needed.

Assumption 3: [2] The triples $(A_{ii},\ B_{ii},\ \sqrt{Q_{ii}})$, $i = 1,\ \dots,N$ are stabilizable and detectable.

III. Asymptotic Structure of CSAREs

Firstly, in order to obtain the strategy that is based on the numerical solutions, the asymptotic structure of the CSAREs (6) is established. Since A_ε, $S_{i\varepsilon}$, $S_{ij\varepsilon}$ and $M_{i\mu}$ include the term of the small parameters ε and μ, the solution $P_{i\varepsilon}$ of the CSAREs (6), if it exists, must contain these parameters. Moreover, it should be noted that two parameters ε and μ are the same magnitude such that Assumption 1 holds. Taking these facts into account, the solution $P_{i\varepsilon}$ of the CSAREs (6) with the following structure is considered [4–6].

$$\begin{aligned}&P_{i\varepsilon}\\ &:= \begin{bmatrix} \varepsilon^{1-\delta_{i1}}P_{i1} & \varepsilon P_{i12} & \cdots & \varepsilon P_{i1N} \\ \varepsilon P_{i12}^T & \varepsilon^{1-\delta_{i2}}P_{i2} & \cdots & \varepsilon P_{i2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon P_{i1N}^T & \varepsilon P_{i2N}^T & \cdots & \varepsilon^{1-\delta_{iN}}P_{iN} \end{bmatrix} \in \mathbf{R}^{\bar{n}\times\bar{n}}.\end{aligned}$$

Substituting the matrices A_ε, $S_{i\varepsilon}$, $S_{ij\varepsilon}$, $M_{i\mu}$, $Q_{i\varepsilon}$ and $P_{i\varepsilon}$ into the CSAREs (6), letting $\varepsilon = 0$ and $\mu = 0$, and partitioning the CSAREs (6), the following reduced-order algebraic Riccati equations (AREs) are obtained, where $\bar{P}_{ii},\ i = 1,\ \dots,N$ be the 0-order solutions of the CSAREs (6) as $\varepsilon = \mu = 0$.

$$\bar{P}_{ii}A_{ii} + A_{ii}^T\bar{P}_{ii} - \bar{P}_{ii}(S_{ii} - M_{ii})\bar{P}_{ii} + Q_{ii} = 0, \quad (9)$$

where $S_{ii} := B_{ii}R_{ii}^{-1}B_{ii}^T$ and $M_{ii} := E_{ii}V_{ii}^{-1}E_{ii}^T$.

In order to guarantee the existence of a positive semidefinite stabilizing solution of the ARE (9), the following condition is assumed.

Assumption 4: The ARE (9) has a positive semidefinite stabilizing solution such that $A_{ii} - S_{ii}\bar{P}_{ii}$ is stable.

The asymptotic expansion of the CSAREs (6) at $\varepsilon = \mu = 0$ is described by the following lemma [7], [8].

Lemma 2: Under Assumptions 1-4, there exist the small constants σ^* and ρ^* such that for all $\varepsilon \in (0,\ \sigma^*)$ and $\mu \in (0,\ \rho^*)$, the CSAREs (6) admits a unique positive semidefinite solution $P_{i\varepsilon}^*$ that can be written as

$$\begin{aligned}P_{i\varepsilon} &:= P_{i\varepsilon}^* = \bar{P}_i + O(\varepsilon)\\ &= \textbf{block diag}\begin{pmatrix} 0 & \cdots & \bar{P}_{ii} & \cdots & 0 \end{pmatrix} + O(\varepsilon). \quad (10)\end{aligned}$$

IV. Newton's Method for Solving CSAREs

In order to obtain the solution of CSAREs (6), the following useful algorithm is given.

$$\begin{aligned}&P_{i\varepsilon}^{(k+1)}\left(A_\varepsilon - \sum_{j=1}^{N} S_{j\varepsilon}P_{j\varepsilon}^{(k)} + M_{i\mu}P_{i\varepsilon}^{(k)}\right)\\ &\quad + \left(A_\varepsilon - \sum_{j=1}^{N} S_{j\varepsilon}P_{j\varepsilon}^{(k)} + M_{i\mu}P_{i\varepsilon}^{(k)}\right)^T P_{i\varepsilon}^{(k+1)}\\ &\quad - \sum_{j=1,\ j\neq i}^{N} P_{j\varepsilon}^{(k+1)}S_{j\varepsilon}P_{i\varepsilon}^{(k)} - \sum_{j=1,\ j\neq i}^{N} P_{i\varepsilon}^{(k)}S_{j\varepsilon}P_{j\varepsilon}^{(k+1)}\\ &\quad + \mu \sum_{j=1,\ j\neq i}^{N} P_{j\varepsilon}^{(k+1)}S_{ij\varepsilon}P_{j\varepsilon}^{(k)} + \mu \sum_{j=1,\ j\neq i}^{N} P_{j\varepsilon}^{(k)}S_{ij\varepsilon}P_{j\varepsilon}^{(k+1)}\\ &\quad + \sum_{j=1,\ j\neq i}^{N} P_{j\varepsilon}^{(k)}S_{j\varepsilon}P_{i\varepsilon}^{(k)} + \sum_{j=1,\ j\neq i}^{N} P_{i\varepsilon}^{(k)}S_{j\varepsilon}P_{j\varepsilon}^{(k)}\\ &\quad + P_{i\varepsilon}^{(k)}S_{i\varepsilon}P_{i\varepsilon}^{(k)} - \mu \sum_{j=1,\ j\neq i}^{N} P_{j\varepsilon}^{(k)}S_{ij\varepsilon}P_{j\varepsilon}^{(k)}\\ &\quad - P_{i\varepsilon}^{(k)}M_{i\mu}P_{i\varepsilon}^{(k)} + Q_{i\varepsilon} = 0,\quad k = 0,\ 1,\ \dots, \quad (11)\end{aligned}$$

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where

$$P_{i\varepsilon}^{(k)} := \begin{bmatrix} \varepsilon^{1-\delta_{i1}} P_{i1}^{(k)} & \varepsilon P_{i12}^{(k)} & \cdots & \varepsilon P_{i1N}^{(k)} \\ \varepsilon P_{i12}^{(k)T} & \varepsilon^{1-\delta_{i2}} P_{i2}^{(k)} & \cdots & \varepsilon P_{i2N}^{(k)} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon P_{i1N}^{(k)T} & \varepsilon P_{i2N}^{(k)T} & \cdots & \varepsilon^{1-\delta_{iN}} P_{iN}^{(k)} \end{bmatrix}$$

with the initial conditions

$$P_{i\varepsilon}^{(0)} = \bar{P}_i = \textbf{block diag} \begin{pmatrix} 0 & \cdots & \bar{P}_{ii} & \cdots & 0 \end{pmatrix}. \quad (12)$$

The following lemma indicates that the proposed algorithm (11) which is based on the Newton's method attains the quadratic convergence [7], [8].

Lemma 3: Under Assumptions 1-4, there exist the small constants $\bar{\sigma}$ and $\bar{\rho}$ such that for all $\varepsilon \in (0,\ \bar{\sigma}),\ \bar{\sigma} \le \sigma^*$ and $\mu \in (0,\ \bar{\rho}),\ \bar{\rho} \le \rho^*$, the iterative algorithm (11) converges to the exact solution of $P_{i\varepsilon}^*$ with the rate of the quadratic convergence, where $P_{i\varepsilon}^{(k)}$ is positive semidefinite matrix and $A_\varepsilon - \sum_{j=1}^{N} S_{j\varepsilon} P_{j\varepsilon}^{(k)} + M_{i\mu} P_{i\varepsilon}^{(k)}$ is stable. Moreover, the convergence solutions attain a local unique solution $P_{i\varepsilon}^*$ of the CSAREs (6) in the neighborhood of the initial condition $P_{i\varepsilon}^{(0)} = \bar{P}_i$. That is, the following condition is satisfied.

$$\|P_{i\varepsilon}^{(k)} - P_{i\varepsilon}^*\| = O(\varepsilon^{2^k}),\ k = 0,\ 1,\ \ldots\ . \quad (13)$$

The following theorem will play an important role in establishing error estimate.

Newton-Kantorovich theorem [9] : *Let X and Y be Banach spaces, D be an open convex subset of X. Let $F : D \subseteq X \to Y$ be Fréche differentiable. Assume that, at some $\mathbf{x}_0 \in D$, $F'(\mathbf{x}_0)$ is invertible and that*

(a) $\|F'(\mathbf{x}_0)^{-1}(F'(\mathbf{x}) - F'(\mathbf{y}))\| \le K\|\mathbf{x} - \mathbf{y}\|,\ \mathbf{x},\ \mathbf{y} \in D$.
(b) $\|F'(\mathbf{x}_0)^{-1}F(\mathbf{x}_0)\| \le \eta,\ h = K\eta \le 1/2$.
(c) $\bar{S}(\mathbf{x}_0,\ t^*) := \{\ \mathbf{x} :\ \ \|\mathbf{x} - \mathbf{x}_0\| \le t^*\} \subseteq D,\ t^* = 2\eta/(1 + \sqrt{1 - 2h})$.

Then:

(i) *The Newton iterations $\mathbf{x}_{k+1} = \mathbf{x}_k - F'(\mathbf{x}_k)^{-1}F(\mathbf{x}_k)$ are well-defined, lie in $\bar{S}(\mathbf{x}_0,\ t^*)$ and converge to a solution $\mathbf{x}^*$ of $F(\mathbf{x}) = 0$.*
(ii) *The solution $\mathbf{x}^*$ is unique in $S(\mathbf{x}_0,\ t^{**}) := \{\ \mathbf{x} :\ \|\mathbf{x} - \mathbf{x}_0\| < t^{**}\} \cap D,\ t^{**} = (1 + \sqrt{1 - 2h})/K$ if $2h < 1$, and in $\bar{S}(\mathbf{x}_0,\ t^{**}) := \{\ \mathbf{x} :\ \ \|\mathbf{x} - \mathbf{x}_0\| \le t^{**}\}$ if $2h = 1$.*
(iii) *Error estimates*

$$\|\mathbf{x}^* - \mathbf{x}_k\| \le \begin{cases} t^* & k = 0 \\ 2^{1-k}(2h)^{2^k - 1}\eta, & k \ge 1 \end{cases},$$

are valid.

Now, let us prove Lemma 3.

Proof: It is immediately obtained from the CSAREs (6) that there exists a positive scalar $\bar{\gamma}$ such that for any $P_{i\varepsilon}^a$ and $P_{i\varepsilon}^b$

$$\begin{aligned} &\|\nabla \boldsymbol{G}(P_{1\varepsilon}^a,\ \ldots,P_{N\varepsilon}^a) - \nabla \boldsymbol{G}(P_{1\varepsilon}^b,\ \ldots,P_{N\varepsilon}^b)\| \\ &\le \bar{\gamma}\|([\mathrm{vec}P_{1\varepsilon}^a]^T,\ \ldots,[\mathrm{vec}P_{N\varepsilon}^a]^T) \\ &\quad -([\mathrm{vec}P_{1\varepsilon}^b]^T,\ \ldots,[\mathrm{vec}P_{N\varepsilon}^b]^T)\|. \end{aligned} \quad (14)$$

Moreover, it is easy to verify that

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_{11}|_{\varepsilon=0} & \cdots & \mathbf{J}_{1N}|_{\varepsilon=0} \\ \vdots & \ddots & \vdots \\ \mathbf{J}_{N1}|_{\varepsilon=0} & \cdots & \mathbf{J}_{NN}|_{\varepsilon=0} \end{bmatrix} = \begin{bmatrix} \boldsymbol{D}_A & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \boldsymbol{D}_A \end{bmatrix} \quad (15)$$

where

$$\begin{aligned} \mathbf{J}_{ij} &= \frac{\partial \mathrm{vec}\boldsymbol{G}_i}{\partial[\mathrm{vec}P_{j\varepsilon}]^T}, \\ \boldsymbol{D}_A &= \textbf{block diag} \begin{pmatrix} \mathbf{D}_{11} & \cdots & \mathbf{D}_{NN} \end{pmatrix}, \\ \mathbf{D}_{ii} &:= D_{ii}^T \otimes I_{n_i} + I_{n_i} \otimes D_{ii}^T, \\ D_{ii} &:= A_{ii} - (S_{ii} - M_{ii})\bar{P}_{ii}. \end{aligned}$$

Thus, since $\mathbf{J}$ is nonsingular under Assumption 4, for small ε and μ, $\nabla \boldsymbol{G}(P_{1\varepsilon}^{(0)},\ \ldots,P_{N\varepsilon}^{(0)}) = \nabla \boldsymbol{G}(\bar{P}_1,\ \ldots,\bar{P}_N) = \mathbf{J} + O(\varepsilon)$ is also nonsingular. Therefore, there exists $\bar{\beta}$ such that $\bar{\beta} = \|[\nabla \boldsymbol{G}(\bar{P}_1,\ \ldots,\bar{P}_N)]^{-1}\|$. On the other hand, since $\|\boldsymbol{G}(\bar{P}_1,\ \ldots,\bar{P}_N)\| = O(\varepsilon)$, there exists $\bar{\eta}$ such that $\bar{\eta} = \|[\nabla \boldsymbol{G}(\bar{P}_1,\ \ldots,\bar{P}_N)]^{-1}\| \cdot \|\boldsymbol{G}(\bar{P}_1,\ \ldots,\bar{P}_N)\| = O(\varepsilon)$. Thus, there exists $\bar{h}$ such that $\bar{h} = \bar{\beta}\bar{\eta}\bar{\gamma} < 2^{-1}$ because $\bar{\eta} = O(\varepsilon)$. Finally, the Newton-Kantorovich theorem results in the desired results (13). On the other hand, since the property of the stability can be proved by using the similar technique [6], it is omitted.

Second, the local uniqueness of the solution is discussed. Now, let us define $\boldsymbol{R}^* \equiv \dfrac{1}{\bar{\gamma}\bar{\beta}}[1 + \sqrt{1 - 2\bar{h}}]$. Clearly, $S \equiv \{\ P_{i\varepsilon}\ :\ \|P_{i\varepsilon} - P_{i\varepsilon}^{(0)}\| \le \boldsymbol{R}^*\ \}$ is in the convex set D. In the sequel, since $\|P_{i\varepsilon} - P_{i\varepsilon}^{(0)}\| = O(\varepsilon)$ holds for a small ε, the local uniqueness of $P_{i\varepsilon}^*$ is guaranteed in the neighbourhood of $\varepsilon = \mu = 0$ for a subset S by applying the Newton-Kantorovich theorem. ■

In order to solve the large-scale Lyapunov equations (11), it should be noted that a fixed-point algorithm and the reduced-order algorithm can be applied (see e.g., [5], [6]).

It is noteworthy that since the proposed design method is based on the Newton's method, the quadratic convergence rate is attained. As a result, the convergence speed is very fast for this algorithm.

V. High-order Soft Constrained Nash Strategy

The required solution of the CSARE (6) exists under Assumptions 1-4. Moreover, it is very important to note that the iterative solutions $P_{i\varepsilon}^{(k)}$ by means of the Newton's method (11) satisfy the positive semidefiniteness, the local uniqueness in the neighbourhood of $\varepsilon = \mu = 0$ and the admissibility. That is, these convergence solutions will satisfy the soft-constrained Nash equilibrium properties (5) for sufficiently small parameters ε and μ.

The attention is focused on the design of the high-order Nash equilibrium strategy for the sign-indefinite linear quadratic games. Such strategy is obtained by using the iterative solution (11).

$$u_i^{(k)*}(t) = -R_{ii}^{-1} B_{i\varepsilon}^T P_{i\varepsilon}^{(k)} x(t),\ i = 1,\ \ldots,N. \quad (16)$$

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The degradation of the cost performance via the new high-order soft constrained Nash equilibrium strategy (16) is given as follows.

Theorem 1: Under Assumptions 1-4, the use of the high-order soft constrained Nash equilibrium strategy (16) results in (17)

$$\begin{aligned}&\bar{J}_i(u_1^{(k)*},\ \dots,\ u_N^{(k)*},\ x(0))\\ &= \bar{J}_i(u_1^*,\ \dots,\ u_N^*,\ x(0)) + O(\varepsilon^{2^k+1}), \qquad (17)\\ &i=1,\ \dots,N.\end{aligned}$$

Proof: When $u_i^{(k)*}(t) = F_{i\varepsilon}^{(k)*}x(t)$ is used, the value of the cost performance is given by

$$\bar{J}_i(u_1^{(k)*},\ \dots,\ u_N^{(k)*},\ x(0)) = x^T(0)Y_{i\varepsilon}x(0), \qquad (18)$$

where $Y_{i\varepsilon}$ is a positive semidefinite solution of the following ARE

$$\begin{aligned}&Y_{i\varepsilon}\left(A_\varepsilon - \sum_{j=1}^N S_{j\varepsilon}P_{j\varepsilon}^{(k)}\right) + \left(A_\varepsilon - \sum_{j=1}^N S_{j\varepsilon}P_{j\varepsilon}^{(k)}\right)^T Y_{i\varepsilon}\\ &+Y_{i\varepsilon}M_{i\mu}Y_{i\varepsilon} + \mu\sum_{j=1,\ j\neq i}^N P_{j\varepsilon}^{(k)}S_{ij\varepsilon}P_{j\varepsilon}^{(k)}\\ &+P_{i\varepsilon}^{(k)}S_{i\varepsilon}P_{i\varepsilon}^{(k)} + Q_{i\varepsilon} = 0. \qquad (19)\end{aligned}$$

Subtracting the CSARE (6) from the ARE (19), $Z_{i\varepsilon} = Y_{i\varepsilon} - P_{i\varepsilon}$ satisfies the following ARE

$$\begin{aligned}&Z_{i\varepsilon}\bar{A}_\varepsilon^{(k)} + \bar{A}_\varepsilon^{(k)T}Z_{i\varepsilon} + Z_{i\varepsilon}M_{i\mu}Z_{i\varepsilon}\\ &+\sum_{j=1,\ j\neq i}^N P_{i\varepsilon}S_{j\varepsilon}\left(P_{j\varepsilon} - P_{j\varepsilon}^{(k)}\right)\\ &+\sum_{j=1,\ j\neq i}^N \left(P_{j\varepsilon} - P_{j\varepsilon}^{(k)}\right)S_{j\varepsilon}P_{i\varepsilon}\\ &+\mu\left[\sum_{j=1,\ j\neq i}^N (P_{j\varepsilon}^{(k)}S_{ij\varepsilon}P_{j\varepsilon}^{(k)} - P_{j\varepsilon}S_{ij\varepsilon}P_{j\varepsilon})\right]\\ &+\left(P_{i\varepsilon} - P_{i\varepsilon}^{(k)}\right)S_{i\varepsilon}\left(P_{i\varepsilon} - P_{i\varepsilon}^{(k)}\right) = 0, \qquad (20)\end{aligned}$$

where $\bar{A}_\varepsilon^{(k)} := A_\varepsilon - \sum_{j=1}^N S_{j\varepsilon}P_{j\varepsilon}^{(k)} + M_{i\mu}P_{i\varepsilon}^{(k)} + M_{i\mu}(P_{i\varepsilon} - P_{i\varepsilon}^{(k)})$.

Taking (13) into account as $P_{i\varepsilon} = P_{i\varepsilon}^*$, it is easy to verify that

$$\begin{aligned}&\boldsymbol{H}(Z_{i\varepsilon}) := Z_{i\varepsilon}\left(\boldsymbol{D}_A + O(\varepsilon)\right) + \left(\boldsymbol{D}_A + O(\varepsilon)\right)^T Z_{i\varepsilon}\\ &\qquad +Z_{i\varepsilon}M_{i\mu}Z_{i\varepsilon} + O(\varepsilon^{2^k+1}) = 0. \qquad (21)\end{aligned}$$

Since the function $\boldsymbol{H}(Z_{i\varepsilon})$ is continuous at any $Z_{i\varepsilon}$, taking the partial derivative of the function $\boldsymbol{H}(Z_{i\varepsilon})$ with respect to $Z_{i\varepsilon}$ yields

$$\begin{aligned}&\nabla\boldsymbol{H}(Z_{i\varepsilon}) := I_{\bar{n}}\otimes(\boldsymbol{D}_A + O(\varepsilon) + M_{i\mu}Z_{i\varepsilon})^T\\ &\qquad +(\boldsymbol{D}_A + O(\varepsilon) + M_{i\mu}Z_{i\varepsilon})^T\otimes I_{\bar{n}}. \qquad (22)\end{aligned}$$

Thus, by using Assumption 4, there exists a small constant $\hat{\sigma}$ such that for all $\varepsilon\in(0,\ \hat{\sigma})$,

$$\nabla\boldsymbol{H}(0) = I_{\bar{n}}\otimes(\boldsymbol{D}_A + O(\varepsilon))^T + (\boldsymbol{D}_A + O(\varepsilon))^T\otimes I_{\bar{n}} \qquad (23)$$

is nonsingular. Then, for any matrices $\boldsymbol{X}_\varepsilon$ and $\boldsymbol{Y}_\varepsilon$ that belong to $Z_{i\varepsilon}$, it is immediately obtained from equation (22) that

$$\|\nabla\boldsymbol{H}(\boldsymbol{X}_\varepsilon) - \nabla\boldsymbol{H}(\boldsymbol{Y}_\varepsilon)\| \le \hat{\gamma}\|\boldsymbol{X}_\varepsilon - \boldsymbol{Y}_\varepsilon\|, \qquad (24)$$

where $\hat{\gamma} := 2\|M_{i\mu}\|$.

Moreover, there exists $\hat{\eta}$ such that

$$\|[\nabla\boldsymbol{H}(0)]^{-1}\boldsymbol{H}(0)\| < O(\varepsilon^{2^k+1}) = \hat{\eta} \qquad (25)$$

because of $\boldsymbol{H}(0) = O(\varepsilon^{2^k+1})$. Using the Newton-Kantorovich theorem, the estimate is given by

$$\|Z_{i\varepsilon} - 0\| = \|Z_{i\varepsilon}\| \le 2\hat{\eta} = O(\varepsilon^{2^k+1}). \qquad (26)$$

Hence

$$\begin{aligned}&x(0)^T Z_{i\varepsilon}x(0) = x(0)^T Y_{i\varepsilon}x(0) - x^T(0)P_{i\varepsilon}x(0)\\ &= \bar{J}_i(u_1^{(k)*},\ \dots,\ u_N^{(k)*},\ x(0))\\ &\quad -\bar{J}_i(u_1^*,\ \dots,\ u_N^*,\ x(0)) = O(\varepsilon^{2^k+1}) \qquad (27)\end{aligned}$$

results in (17). ■

It should be noted that the present proof is quite different from the existing one [8] because it is based on Newton-Kantorovich theorem. As a result, it seems to be a novel contribution because the obtained results can be proved directly without using the transformation.

Using the similar technique of the proof of Theorem 1, the following conditions are satisfied.

Theorem 2: Under Assumptions 1-4, the following result holds.

$$\begin{aligned}&\bar{J}_i(u_1^{(k)*},\ \dots,\ u_i,\ \dots,\ u_N^{(k)*},\ x(0))\\ &= \bar{J}_i(u_1^*,\ \dots,\ u_i,\ \dots,\ u_N^*,\ x(0)) + O(\varepsilon^{2^k+1}), \qquad (28)\\ &i=1,\ \dots,N.\end{aligned}$$

Proof: Since this proof can be done by using the similar technique used in Theorem 1, it is omitted. ■

Finally, the main result is easily derived.

Theorem 3: Under Assumptions 1-4, the use of the high-order soft constrained Nash strategies (16) results in

$$\begin{aligned}&\bar{J}_i(u_1^{(k)*},\ \dots,\ u_N^{(k)*},\ x(0))\\ &\le \bar{J}_i(u_1^{(k)*},\ \dots,\ u_i,\ \dots,\ u_N^{(k)*},\ x(0)) + O(\varepsilon^{2^k+1}), \qquad (29)\\ &i=1,\ \dots,N.\end{aligned}$$

Proof: Let us introduce the following equality.

$$\begin{aligned}&\bar{J}_i(u_1^{(k)*},\ \dots,\ u_N^{(k)*})\\ &\quad -\bar{J}_i(u_1^{(k)*},\ \dots,\ u_{i-1}^{(k)*},\ u_i,\ u_{i+1}^{(k)*},\ \dots,\ u_N^{(k)*})\\ &= \bar{J}_i(u_1^{(k)*},\ \dots,\ u_N^{(k)*}) - \bar{J}_i(u_1^*,\ \dots,\ u_N^*)\\ &\quad +\bar{J}_i(u_1^*,\ \dots,\ u_N^*)\\ &\quad -\bar{J}_i(u_1^*,\ \dots,\ u_{i-1}^*,\ u_i,\ u_{i+1}^*,\ \dots,\ u_N^*)\\ &\quad +\bar{J}_i(u_1^*,\ \dots,\ u_{i-1}^*,\ u_i,\ u_{i+1}^*,\ \dots,\ u_N^*)\\ &\quad -\bar{J}_i(u_1^{(k)*},\ \dots,\ u_{i-1}^{(k)*},\ u_i,\ u_{i+1}^{(k)*},\ \dots,\ u_N^{(k)*}). \qquad (30)\end{aligned}$$

Using (5), (17) and (28), the proof of (29) can be completed. The other case is similar. ■

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VI. RECURSIVE ALGORITHM

A. Computational Algorithm

In order to reduce amount of computation and CPU time, a new algorithm for solving CSAREs (6) that is based on the recursive algorithm is established. In particular, the following special case of $N = 2$ is considered because it is easy to extend it to the general case. Furthermore, without loss of generality, in order to simplify the algebra, it is assumed that $\mu = 0$. On the other hand, it may be noted that we are studying a more general case than the one studied in [10]. That is, it is worth pointing out that since $M_{i\mu}$ exists, ARE (9) has indefinite sign.

In order to simplify the notation, let us define the following matrix.

$$S_{i\varepsilon} - M_{i\mu} := \begin{bmatrix} U_{i1} + O(\varepsilon^2) & \varepsilon U_{i12} \\ \varepsilon U_{i12}^T & U_{i2} + O(\varepsilon^2) \end{bmatrix}. \quad (31)$$

The solutions of CSAREs (6) can be expressed as follows.

$$P_{ii} = \bar{P}_{ii} + \varepsilon E_{ii},\ P_{ij} = \bar{P}_{ij} + \varepsilon E_{ij},\ i \neq j,$$
$$P_{i12} = \bar{P}_{i12} + \varepsilon E_{i12}, \quad (32)$$

where $i,\ j = 1,\ 2$ and $U_{ii} := S_{ii} - M_{ii}$,

$$\bar{P}_{ii}A_{ii} + A_{ii}^T\bar{P}_{ii} - \bar{P}_{ii}U_{ii}\bar{P}_{ii} + Q_{ii} = 0,$$
$$\bar{P}_{ij}D_{jj} + D_{jj}^T\bar{P}_{ij} + Q_{ij} = 0,\ i \neq j,$$
$$\bar{P}_{112}D_{22} + D_{11}^T\bar{P}_{112} + \bar{P}_{11}A_{12} - \bar{P}_{11}U_{212}\bar{P}_{22} + Q_{112} = 0,$$
$$\bar{P}_{212}D_{22} + D_{11}^T\bar{P}_{212} + A_{21}^T\bar{P}_{22} - \bar{P}_{11}U_{112}\bar{P}_{22} + Q_{212} = 0.$$

$\bar{P}_{ii}$, $\bar{P}_{ij}$ and $\bar{P}_{i12}$ are the first order approximations of $P_{i\varepsilon}$ corresponding to ε. It should be noted that the $O(\varepsilon^2)$ approximation of the original CSAREs (6) has been considered in [10]. It means that another recursive algorithm for obtaining solutions $P_{ij}(\varepsilon)$ and $P_{i12}(\varepsilon)$ as $O(\varepsilon^2)$ has to be needed. On the other hand, since the $O(\varepsilon)$ approximation is treated in this paper, these solutions can be obtained independently for small ε. That is, by using the function `ARE` of MATLAB, these solutions can be computed with high-order precision without any other algorithm.

Substituting the solutions of (32) into CSAREs (6) and after some tedious algebra, the following expressions for the error equations (33) are obtained.

$$E_{ii}D_{ii} + D_{ii}^TE_{ii} + h_i = 0,\ i = 1,\ 2, \quad (33a)$$
$$E_{ij}D_{jj} + D_{jj}^TE_{ij} - E_{jj}U_{jj}\bar{P}_{ij} - \bar{P}_{ij}U_{jj}E_{jj}$$
$$+h_l = 0,\ i \neq j,\ l = 3,\ 4, \quad (33b)$$
$$E_{212}D_{22} + D_{11}^TE_{212}$$
$$+(A_{21} - U_{112}^T\bar{P}_{11} - U_{22}\bar{P}_{212}^T)^TE_{22}$$
$$-E_{11}(U_{112}\bar{P}_{22} + U_{11}\bar{P}_{212}) + h_5 = 0, \quad (33c)$$
$$E_{112}D_{22} + D_{11}^TE_{112}$$
$$+E_{11}(A_{12} - U_{11}\bar{P}_{112} - U_{212}\bar{P}_{22})$$
$$-(U_{212}^T\bar{P}_{11} + U_{22}\bar{P}_{112}^T)^TE_{22} + h_6 = 0, \quad (33d)$$

where $h_l := h_l(\varepsilon,\ \varepsilon E_{11},\ \dots,\varepsilon E_{22})$, $l = 1,\ \dots,6$.

Hence, the following recursive algorithm for solving CSAREs (6) is given at the top of the next page.

The main result of this section is given below.

Theorem 4: Under Assumptions 1-4, the recursive algorithm (34) converges to the exact solutions E_{ij} and E_{i12}, $i,\ j = 1,\ 2$ of equation (33) with the rate of the linear convergence. That is, the following conditions are satisfied.

$$\|E_{ij} - E_{ij}^{(n)}\| = O(\varepsilon^n),\ \|E_{i12} - E_{i12}^{(n)}\| = O(\varepsilon^n),$$
$$n = 1,\ 2,\ \dots,\ i,\ j = 1,\ 2. \quad (35)$$

Proof: The proof of (35) uses mathematical induction. When $n = 0$ for the equations (34), the first order approximations $E_{ij}^{(1)}$ and $E_{i12}^{(1)}$ corresponding to the small parameter ε satisfy the equations (33). It follows from these equations that

$$\|E_{ij} - E_{ij}^{(1)}\| = O(\varepsilon),$$
$$\|E_{i12} - E_{i12}^{(1)}\| = O(\varepsilon),\quad i,\ j = 1,\ 2. \quad (36)$$

When $n = m,\ m \geq 1$, it is assumed that $\|E_{ij} - E_{ij}^{(m)}\| = O(\varepsilon^m)$ and $\|E_{i12} - E_{i12}^{(m)}\| = O(\varepsilon^m)$. Subtracting (33) from (34) and using the assumptions $\|E_{ij} - E_{ij}^{(m)}\| = O(\varepsilon^m)$ and $\|E_{i12} - E_{i12}^{(m)}\| = O(\varepsilon^m)$, since D_{ii} $i = 1,\ 2$ are stable, using Assumption 2 the following results hold.

$$\|E_{ij} - E_{ij}^{(m+1)}\| = O(\varepsilon^{m+1}),$$
$$\|E_{i12} - E_{i12}^{(m+1)}\| = O(\varepsilon^{m+1}). \quad (37)$$

Consequently, equation (35) holds for all $n \in \mathbf{N}$. This completes the proof of Theorem 4 concerned with the recursive algorithm. ■

The required iterative count associated with the Newton's method with other two fixed point algorithms [7], [8] and the proposed recursive algorithm is compared. It is assumed that the required operation count is $O(n)$ by using the recursive algorithm for rough estimate. In this case, it should be noted that the required operation count of these fixed point algorithms are also $O(n)$, respectively. Then, the required operation count of the Newton's method with other two fixed point algorithms is $O(n^2 \log_2 n)$. Therefore, the proposed recursive algorithm drastically succeeds in reducing the operating count. As a result, the CPU time can be reduced.

B. Performance Degradation

In addition, it will be presented an important implication. If the recursive solutions are considered instead of the Newton's iterative solutions, then the following corollaries are easily seen in view of Theorems 1-3.

Let us consider the following strategy set that is obtained by using the iterative solution (34).

$$u_i^{(n)*}(t) = -R_{ii}^{-1}B_{i\varepsilon}^TP_{i\varepsilon}^{(n)}x(t),\ i = 1,\ \dots,N, \quad (38)$$

where $u_i^*(t) = u_i^{(n)*}(t) + O(\varepsilon)$.

The degradation of the cost performance via the new high-order soft constrained Nash equilibrium strategy (38) is given as follows.

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$$
\begin{aligned}
& E_{11}^{(n+1)} D_{11} + D_{11}^T E_{11}^{(n+1)} \\
&= -\varepsilon(P_{112}^{(n)} A_{21} + A_{21}^T P_{112}^{(n)T}) + \varepsilon(P_{11}^{(n)} U_{212} P_{212}^{(n)T} + P_{212}^{(n)} U_{212}^T P_{11}^{(n)} + P_{112}^{(n)} U_{22} P_{212}^{(n)T} + P_{212}^{(n)} U_{22} P_{112}^{(n)T}) \\
&\quad +\varepsilon^2(P_{11}^{(n)} U_{211} P_{21}^{(n)} + P_{21}^{(n)} U_{211} P_{11}^{(n)} + P_{112}^{(n)} U_{212}^T P_{21}^{(n)} + P_{21}^{(n)} U_{212} P_{112}^{(n)T}) + \varepsilon E_{11}^{(n)} U_{11} E_{11}^{(n)} \\
&\quad +\varepsilon(P_{112}^{(n)} U_{112}^T P_{11}^{(n)} + P_{11}^{(n)} U_{112} P_{112}^{(n)T}) + \varepsilon^3 P_{112}^{(n)} U_{122} P_{112}^{(n)T}, \qquad \text{(34a)} \\
& E_{22}^{(n+1)} D_{22} + D_{22}^T E_{22}^{(n+1)} \\
&= -\varepsilon(P_{212}^{(n)T} A_{12} + A_{12}^T P_{212}^{(n)}) + \varepsilon(P_{112}^{(n)T} U_{11} P_{212}^{(n)} + P_{212}^{(n)T} U_{11} P_{112}^{(n)} + P_{112}^{(n)T} U_{112} P_{22}^{(n)} + P_{22}^{(n)} U_{112}^T P_{112}^{(n)}) \\
&\quad +\varepsilon^2(P_{12}^{(n)} U_{112}^T P_{212}^{(n)} + P_{212}^{(n)T} U_{112} P_{12}^{(n)} + P_{12}^{(n)} U_{122} P_{22}^{(n)} + P_{22}^{(n)} U_{122} P_{12}^{(n)}) + \varepsilon E_{22}^{(n)} U_{22} E_{22}^{(n)} \\
&\quad +\varepsilon(P_{22}^{(n)} U_{212}^T P_{212}^{(n)} + P_{212}^{(n)T} U_{212} P_{22}^{(n)}) + \varepsilon^3 P_{212}^{(n)T} U_{211} P_{212}^{(n)}, \qquad \text{(34b)} \\
& E_{12}^{(n+1)} D_{22} + D_{22}^T E_{12}^{(n+1)} \\
&= E_{22}^{(n+1)} U_{22} \bar{P}_{12} + \bar{P}_{12} U_{22} E_{22}^{(n+1)} + \varepsilon(E_{12}^{(n)} U_{22} E_{22}^{(n)} + E_{22}^{(n)} U_{22} E_{12}^{(n)}) \\
&\quad -(P_{112}^{(n)T} A_{12} + A_{12}^T P_{112}^{(n)}) + P_{112}^{(n)T} U_{212} P_{22}^{(n)} + P_{22}^{(n)} U_{212}^T P_{112}^{(n)} + \varepsilon^2(P_{112}^{(n)T} U_{211} P_{212}^{(n)} + P_{212}^{(n)T} U_{211} P_{112}^{(n)}) \\
&\quad +\varepsilon(P_{12}^{(n)} U_{212}^T P_{212}^{(n)} + P_{212}^{(n)T} U_{212} P_{12}^{(n)}) + P_{112}^{(n)T} U_{11} P_{112}^{(n)} + \varepsilon^2 P_{12}^{(n)} U_{122} P_{12}^{(n)} \\
&\quad +\varepsilon(P_{12}^{(n)} U_{112}^T P_{112}^{(n)} + P_{112}^{(n)T} U_{112} P_{12}^{(n)}), \qquad \text{(34c)} \\
& E_{21}^{(n+1)} D_{11} + D_{11}^T E_{21}^{(n+1)} \\
&= E_{11}^{(n+1)} U_{11} \bar{P}_{21} + \bar{P}_{21} U_{11} E_{11}^{(n+1)} + \varepsilon(E_{21}^{(n)} U_{11} E_{11}^{(n)} + E_{11}^{(n)} U_{11} E_{21}^{(n)}) \\
&\quad -(P_{212}^{(n)} A_{21} + A_{21}^T P_{212}^{(n)T}) + P_{212}^{(n)} U_{112}^T P_{11}^{(n)} + P_{11}^{(n)} U_{112} P_{212}^{(n)T} + \varepsilon^2(P_{212}^{(n)} U_{122} P_{112}^{(n)T} + P_{112}^{(n)} U_{122} P_{212}^{(n)T}) \\
&\quad +\varepsilon(P_{21}^{(n)} U_{112} P_{112}^{(n)T} + P_{112}^{(n)} U_{112}^T P_{21}^{(n)}) + P_{212}^{(n)} U_{22} P_{212}^{(n)T} + \varepsilon^2 P_{21}^{(n)} U_{211} P_{21}^{(n)} \\
&\quad +\varepsilon(P_{212}^{(n)} U_{212}^T P_{21}^{(n)} + P_{21}^{(n)} U_{212} P_{212}^{(n)T}), \qquad \text{(34d)} \\
& E_{212}^{(n+1)} D_{22} + D_{11}^T E_{212}^{(n+1)} \\
&= -(A_{21} - U_{112}^T \bar{P}_{11} - U_{22} \bar{P}_{212}^T)^T E_{22}^{(n+1)} + E_{11}^{(n+1)}(U_{112} \bar{P}_{22} + U_{11} \bar{P}_{212}) \\
&\quad +\varepsilon(E_{11}^{(n)} U_{112} E_{22}^{(n)} + E_{11}^{(n)} U_{11} E_{212}^{(n)} + E_{212}^{(n)} U_{22} E_{22}^{(n)}) - P_{21}^{(n)} A_{12} \\
&\quad +\varepsilon(P_{21}^{(n)} U_{112} P_{12}^{(n)} + P_{212}^{(n)} U_{112}^T P_{112}^{(n)} + P_{112}^{(n)} U_{112}^T P_{212}^{(n)} + P_{112}^{(n)} U_{122} P_{22}^{(n)}) \\
&\quad +\varepsilon^2 P_{212}^{(n)} U_{122} P_{12}^{(n)} + P_{21}^{(n)} U_{11} P_{112}^{(n)} + \varepsilon P_{212}^{(n)} U_{212}^T P_{212}^{(n)} + P_{21}^{(n)} U_{212} P_{22}^{(n)} + \varepsilon^2 P_{21}^{(n)} U_{211} P_{212}^{(n)}, \qquad \text{(34e)} \\
& E_{112}^{(n+1)} D_{22} + D_{11}^T E_{112}^{(n+1)} \\
&= -E_{11}^{(n+1)}(A_{12} - U_{11} \bar{P}_{112} - U_{212} \bar{P}_{22}) + (U_{212}^T \bar{P}_{11} + U_{22} \bar{P}_{112}^T)^T E_{22}^{(n+1)} \\
&\quad +\varepsilon(E_{11}^{(n)} U_{212} E_{22}^{(n)} + E_{112}^{(n)} U_{22} E_{22}^{(n)} + E_{11}^{(n)} U_{11} E_{112}^{(n)}) - A_{21}^T P_{12}^{(n)} \\
&\quad +\varepsilon(P_{21}^{(n)} U_{212} P_{12}^{(n)} + P_{11}^{(n)} U_{211} P_{212}^{(n)} + P_{112}^{(n)} U_{212}^T P_{212}^{(n)} + P_{212}^{(n)} U_{212}^T P_{112}^{(n)}) \\
&\quad +\varepsilon^2 P_{21}^{(n)} U_{211} P_{112}^{(n)} + P_{212}^{(n)} U_{22} P_{12}^{(n)} + \varepsilon P_{112}^{(n)} U_{112}^T P_{112}^{(n)} + P_{11}^{(n)} U_{112} P_{12}^{(n)} + \varepsilon^2 P_{112}^{(n)} U_{122} P_{12}^{(n)}, \qquad \text{(34f)}
\end{aligned}
$$

where $n = 0,\ 1,\ 2,\ \dots,\ P_{ij}^{(n)} = \bar{P}_{ij} + \varepsilon E_{ij}^{(n)},\ P_{i12}^{(n)} = \bar{P}_{i12} + \varepsilon E_{i12}^{(n)},\ E_{ij}^{(0)} = 0,\ E_{i12}^{(0)} = 0,\ i,\ j = 1,\ 2.$

Corollary 1: Under Assumptions 1-4, the use of the high-order soft constrained Nash equilibrium strategy (38) results in (39)

$$
\begin{aligned}
& \bar{J}_i(u_1^{(n)*},\ \dots,\ u_N^{(n)*},\ x(0)) \\
&= \bar{J}_i(u_1^*,\ \dots,\ u_N^*,\ x(0)) + O(\varepsilon^{n+2}), \qquad (39) \\
& i = 1,\ \dots,N.
\end{aligned}
$$

Corollary 2: Under Assumptions 1-4, the following result holds.

$$
\begin{aligned}
& \bar{J}_i(u_1^{(n)*},\ \dots,\ u_i,\ \dots,\ u_N^{(n)*},\ x(0)) \\
&= \bar{J}_i(u_1^*,\ \dots,\ u_i,\ \dots,\ u_N^*,\ x(0)) + O(\varepsilon^{n+2}), \quad (40) \\
& i = 1,\ \dots,N.
\end{aligned}
$$

Corollary 3: Under Assumptions 1-4, the use of the high-order soft constrained Nash strategies (38) results in

$$
\begin{aligned}
& \bar{J}_i(u_1^{(n)*},\ \dots,\ u_N^{(n)*},\ x(0)) \\
&\le \bar{J}_i(u_1^{(n)*},\ \dots,\ u_i,\ \dots,\ u_N^{(n)*},\ x(0)) + O(\varepsilon^{n+2}), \quad (41) \\
& i = 1,\ \dots,N.
\end{aligned}
$$

Proof: Since the results of Corollaries 1-3 can be proved by using the similar technique in Theorems 1-3, the proof is omitted. ■

VII. NUMERICAL EXAMPLE

In order to demonstrate the efficiency of the proposed algorithm, a numerical example is given. The system matrices are given at the top of the next page. It may be noted that the considered large-scale systems are given as a modification of the power systems [3].

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$$A_{11} = \begin{bmatrix} 0 & 1 & -0.266 & -0.009 \\ -2.75 & -2.78 & -1.36 & -0.037 \\ 0 & 0 & 0 & 1 \\ -4.95 & 0 & -55.5 & -0.039 \end{bmatrix},\ \varepsilon A_{12} = \begin{bmatrix} 0.0024 & 0 & -0.087 & 0.002 \\ -0.185 & 0 & 1.11 & -0.011 \\ 0 & 0 & 0 & 0 \\ 0.222 & 0 & 8.17 & 0.004 \end{bmatrix},$$

$$\varepsilon A_{13} = \begin{bmatrix} 0.073 & 0 & -0.25 & 0.003 \\ -0.46 & 0 & 2.8 & -0.02 \\ 0 & 0 & 0 & 0 \\ 0.924 & 0 & 17.5 & 0.02 \end{bmatrix},\ \varepsilon A_{21} = \begin{bmatrix} 0.021 & 0 & 0.121 & 0.003 \\ -1.1 & 0 & -1.62 & -0.015 \\ 0 & 0 & 0 & 0 \\ -2.43 & 0 & 1.37 & -0.034 \end{bmatrix},$$

$$A_{22} = \begin{bmatrix} -0.21 & 1 & -1.6 & -0.005 \\ -1.9 & -1.8 & 9.3 & -0.12 \\ 0 & 0 & 0 & 1 \\ -3.1 & 0 & -56 & 0.032 \end{bmatrix},\ \varepsilon A_{23} = \begin{bmatrix} 0.06 & 0 & 0.46 & 0.002 \\ -1 & 0 & 1.49 & -0.04 \\ 0 & 0 & 0 & 0 \\ 0.12 & 0 & 29.8 & -0.028 \end{bmatrix},$$

$$\varepsilon A_{31} = \begin{bmatrix} -0.002 & 0 & 0.83 & 0 \\ -6.78 & 0 & -10.1 & 0.09 \\ 0 & 0 & 0 & 0 \\ -1.24 & 0 & 0.498 & -0.017 \end{bmatrix},\ \varepsilon A_{32} = \begin{bmatrix} 0.011 & 0 & 0.22 & 0 \\ -2.1 & 0 & 1.7 & -0.123 \\ 0 & 0 & 0 & 0 \\ -0.07 & 0 & 6.38 & -0.011 \end{bmatrix},$$

$$A_{33} = \begin{bmatrix} -0.197 & 1 & -1.2 & -0.003 \\ -54.5 & -20 & 70.1 & -2.37 \\ 0 & 0 & 0 & 1 \\ -3.4 & 0 & -21.0 & -0.017 \end{bmatrix},\ B_{11} = \begin{bmatrix} 0 \\ 36.1 \\ 0 \\ 0 \end{bmatrix},\ B_{22} = \begin{bmatrix} 0 \\ 78.9 \\ 0 \\ 0 \end{bmatrix},\ B_{33} = \begin{bmatrix} 0 \\ 1000 \\ 0 \\ 0 \end{bmatrix},\ B_{ij} = 0,\ i \neq j,$$

$$E_{11} = \begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1 \\ 0 & 0 & 0 & 0.1 \end{bmatrix},\ E_{22} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1 \\ 0 & 0 & 0 & 0.1 \end{bmatrix},\ E_{33} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.1 & 0 & 0 & 0.1 \\ 0 & 0 & 0 & 0.1 \end{bmatrix},\ E_{ij} = 0,\ i \neq j,$$

$$V_{ii} = \mathbf{diag}\,(\ 1\ \ 2\ \ 2\ \ 1\ ),\ V_1 = \mathbf{block\ diag}\,(\ V_{ii}\ \ \mu^{-1}I_4\ \ \mu^{-1}I_4\ ),$$
$$V_2 = \mathbf{block\ diag}\,(\ \mu^{-1}I_4\ \ V_{ii}\ \ \mu^{-1}I_4\ ),\ V_3 = \mathbf{block\ diag}\,(\ \mu^{-1}I_4\ \ \mu^{-1}I_4\ \ V_{ii}\ ),$$
$$Q_1 = \mathbf{block\ diag}\,(\ 0.5I_4\ \ O_{8\times 8}\ ),\ Q_2 = \mathbf{block\ diag}\,(\ O_{4\times 4}\ \ 0.5I_4\ \ O_{4\times 4}\ ),\ Q_3 = \mathbf{block\ diag}\,(\ O_{8\times 8}\ \ 0.5I_4\ ),$$
$$R_{11} = R_{22} = R_{33} = 1,\ R_{12} = R_{13} = 0.2,\ R_{23} = R_{21} = 0.3,\ R_{31} = R_{32} = 0.1.$$

$$F_{1\varepsilon}^{(3)*} = [\ -1.6070\ \ -9.6942e-01\ \ -7.2616\ \ -2.0448e-02\ \ -1.0183e-03\ \ -8.1937e-06$$
$$-3.6317e-03\ \ 9.9899e-03\ \ 2.5572e-03\ \ 2.5593e-06\ \ 3.5754e-02\ \ 1.5000e-03\ ],$$
$$F_{2\varepsilon}^{(3)*} = [\ -1.4029e-03\ \ -3.3082e-05\ \ 2.9290e-02\ \ 1.4973e-03\ \ -1.2967\ \ -9.9375e-01$$
$$-7.9714\ \ 9.9917e-02\ \ 3.5550e-03\ \ 3.5495e-06\ \ 6.6756e-02\ \ 9.0373e-04\ ],$$
$$F_{3\varepsilon}^{(3)*} = [\ -2.2157e-03\ \ -5.8545e-05\ \ -9.9676e-06\ \ 2.4311e-04\ \ -1.1642e-03\ \ -1.4511e-05$$
$$-5.4604e-03\ \ 8.6321e-04\ \ 1.5201\ \ -9.8172e-01\ \ -3.5409\ \ 3.0341e-01\ ].$$

The small parameters are chosen as $\varepsilon = 0.01$ and $\mu = 0.005$.

It is easy to verify that algorithm (11) converges to the exact solution with an accuracy of $\|\boldsymbol{G}^{(k)}\| < 1.0e-11$ after three iterations, where $\|\boldsymbol{G}^{(k)}\| := \sum_{i=1}^{3} \|\boldsymbol{G}_i(\ P_{1\varepsilon}^{(k)},\ P_{2\varepsilon}^{(k)},\ P_{3\varepsilon}^{(k)})\|$.

Table 1. Errors per iterations.

k	$\|\boldsymbol{G}^{(k)}\|$
0	$1.1923e-01$
1	$6.9726e-04$
2	$3.1597e-08$
3	$3.1552e-11$

In order to verify the exactitude of the solution, the remainder per iteration is substituted by $P_{i\varepsilon}^{(k)}$ into CSAREs (6). In Table 1, the results of the error $\|\boldsymbol{G}^{(k)}\|$ per iteration are given. As a result, it can be seen that algorithm (11) yields quadratic convergence.

Using the proposed design procedure, the high-order approximate soft-constrained Nash strategies (16) are given by

$$u_i^{(3)*}(t) = F_{i\varepsilon}^{(3)*}x(t), \tag{42}$$

where

$$F_{1\varepsilon}^{(3)*} = \begin{bmatrix} F_{11} \\ 0 \\ 0 \end{bmatrix},\ F_{2\varepsilon}^{(3)*} = \begin{bmatrix} 0 \\ F_{22} \\ 0 \end{bmatrix},\ F_{3\varepsilon}^{(3)*} = \begin{bmatrix} 0 \\ 0 \\ F_{33} \end{bmatrix}$$

and F_{ii}, $i = 1,\ 2,\ 3$ are given at the top of the next page.

The costs using the high-order strategy (16) are computed. The initial conditions are chosen as $x(0) = [\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1\ ]^T$. The cost functional-to-perturbation ε^{2^k+1} ratio are given in Table 2, where

$$\phi_i = \frac{|\bar{J}_{i\mathrm{app}} - \bar{J}_{i\mathrm{opt}}|}{\varepsilon^{2^k+1}}, \tag{43}$$

$$\bar{J}_{i\mathrm{app}} := \bar{J}_i(u_1^{(k)*},\ \dots,\ u_N^{(k)*},\ x(0)) = x(0)^T Y_{i\varepsilon} x(0),$$
$$\bar{J}_{i\mathrm{opt}} := \bar{J}_i(u_1^*,\ \dots,\ u_N^*,\ x(0)) = x^T(0) P_{i\varepsilon} x(0).$$

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Table 2. Degradations of cost.

k	ϕ_1	ϕ_2	ϕ_3
0	5.8309	1.2554	1.1939
1	$4.8204e-01$	1.7049	4.0481
2	$9.0753e+03$	$2.7508e+04$	$3.9519e+04$
3	$9.0753e+11$	$2.7508e+12$	$3.9519e+12$

It is easy to verify that $\bar{J}_i(u_1^{(k)*},\ \dots\ ,\ u_N^{(k)*},\ x(0)) - \bar{J}_i(u_1^*,\ \dots\ ,\ u_N^*,\ x(0)) = O(\varepsilon^{2^k+1})$ because of $\phi_i < \infty$. It should be noted that the result of three iteration in Table 2 is not reliable because this result has the major error due to $O(\varepsilon^9) \approx 10^{-18}$.

VIII. CONCLUSION

In this paper, a higher-order approximate soft constrained Nash strategy for weakly coupled large-scale systems has been proposed. Comparing with the existing result [5–8], there exist three useful and reliable contributions. First, the considered CSAREs have the sign-indefinite quadratic term. Thus, it succeeds in expanding the results for the general case. Second, in order to reduce the amount of the algebraic computation, the recursive algorithm was considered. This algorithm would result in reducing the CPU time. Third, by using these iterative solutions by means of the proposed two algorithm, the high-order approximate strategies have been proposed. As a result, it has been shown that the proposed strategy results in better performance. Finally, the numerical example has shown the excellent results.

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Muneomi Sagara received his B.S. degree in Education from Hiroshima University, Japan, in 2005 and his master degree in Education form Hiroshima University, Japan, in 2007. He is currently Ph.D. candidate at Graduate School of Education, Hiroshima University. His current research interests include stochastic control systems and its application.

Hiroaki Mukaidani received his B.S. degree in integrated arts and sciences from Hiroshima University, Japan, in 1992 and his M.Eng. and Dr.Eng. degrees in information engineering form Hiroshima University, Japan, in 1994 and 1997, respectively. He worked with Hiroshima City University as a Research Associate from 1998 to 2002. Since 2002, he has been with the Graduate School of Education, Hiroshima University, Japan, as an Assistant Professor and currently as an Associate Professor. His current research interests include robust control, dynamic game and its application of singularly perturbed systems and large-scale systems. He is a member of the Institute of Electrical and Electronics Engineers (IEEE).

Toru Yamamoto received his B.Eng. and M.Eng. degrees from the University of Tokushima in 1984 and 1987, respectively, and his D.Eng. degree from Osaka University in 1994. He is a Professor in the Department of Technology and Information Education at Hiroshima University. He was a Visiting Researcher in the Department of Mathematical Engineering and Information Physics at the University of Tokyo in 1991, and during the autumn of 1996, he was a Visiting Professor in the Department of Chemical and Materials Engineering at University of Alberta, Edmonton, Canada. Furthermore, he was a JSPS (Japan Society for the Promotion of Science) research fellow in the University of Alberta in 2006. His current research interests are in the area of selftuning control, intelligent control and process control. He is currently Associate Editor of IEEJ Transactions on Electronics, Information and Systems. Dr. Yamamoto won the Best Poster Paper Award at the 3rd Asian Control Conference in 2000, and received the Prize of Progress from the Institute of Electrical Engineers in Japan (IEEJ) and Technology Award from the Society of Instrumentation and Control Engineers in Japan (SICE) in 2003.

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