

# Fuzzy Probist Reliability of a Communication Network Using Unified Fuzzy Markov Model

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**Abstract**—In this paper a communication network is modeled as a unified fuzzy Markov model. A method is addressed to find its fuzzy probist reliability through fuzzy probist fault tree whose basic events are the failure of each of the states of UFMM. And we apply this method to estimate the fuzzy probist reliability of a web page 'Departments' of the web site [www.ssn.edu.in](http://www.ssn.edu.in).

**Index Terms**—fuzzy probist reliability, communication network, unified fuzzy Markov model (UFMM), fuzzy probist fault tree (FFT), restricted fuzzy arithmetic, linear programming

## I. INTRODUCTION

In this paper, we propose a communication network with the following assumptions whose behavior is characterized in the context of fuzzy probability measures with binary state assumption. And the fuzzy probability measures are represented by triangular fuzzy numbers on [0,1].

- The system consists of various components such that each component is acted as a sub system with different states and each sub system has an initial state from which the control is initiated.
- The control flow between the components and within the component follow Markovian property.
- The process of transition continues till it reaches a state called the terminal state. This ensures the success of the transitions. Hence the system and each component has a terminal state.
- Each component of the system is acted as an initial state.

The reliability analysis of this system belongs to fuzzy probist reliability theory [2, 8]. Here onwards, the term *fuzzy reliability* represents the *fuzzy probist reliability*. And we model this communication network as unified fuzzy Markov model (UFMM), a set of hierarchical fuzzy Markov chains (FMCs) (the top level FMC, the lower level FMC) such that the whole network is the top level fuzzy Markov chain with set of all components as its state space and each state (component) is associated to a lower level FMC. This paper introduces a method to find the fuzzy reliability of the UFMM using fuzzy probist fault tree (FFT) whose basic events are the failure of each of the states of TFMC and the fuzzy probability for the basic events are nothing but the fuzzy probability

of ever reaching an absorbing state failure 'F' from each of the states of TFMC.

This paper is constructed as follows. Section two recalls some basic definitions. In section three, a method is described to find the fuzzy reliability of a fuzzy Markov

chain. And the fuzzy reliability of the UFMM is obtained in section four. In five, the proposed method is illustrated through a real time example. Finally, we end up with some conclusions.

## II. PRELIMINARIES

Here we recall some definitions which are going to be utilized in this work.

### A. Imperfect Markov Chain [6]

Imperfect Markov chain of a Markov chain with state space  $\{1, 2, \dots, n, S\}$  and with the transition probability  $p_{ij}$ , is a Markov chain with state space  $\{1, 2, \dots, n, S, F\}$  of  $n+2$  states in which  $1, 2, \dots, n$  are transient states and  $S, F$  the success and the failure states respectively are the absorbing states. And the elements of its transition probability matrix  $Q$  are defined as,

$$q_{ij} = r_i p_{ij} \text{ for } 1, 2, \dots, n \text{ and } j = 1, 2, \dots, n, S$$

$$q_{iF} = 1 - r_i \text{ for } 1, 2, \dots, n$$

$$q_{SS} = q_{FF} = 1$$

$$q_{Sj} = q_{Fj} = 0 \text{ for } j = 1, 2, \dots, n.$$

where  $r_i$  is the reliability of the state 'i'.

### B. Fuzzy Markov Chain [1]

A fuzzy Markov chain is similar to the discrete time, discrete state space  $U = \{1, 2, \dots, n\}$  stochastic Markov chain with the transition fuzzy probability  $\tilde{p}_{ij}$  of control transfers from state 'i' to state 'j',  $i, j \in U$ .

### C. Fuzzy Reliability or Fuzzy Probist Reliability [2, 8]

The fuzzy reliability of the proposed system is defined to be the fuzzy probability that the system eventually completes its task successfully without failing from transition from one state to other state till its termination state is reached.

### D. Unified Fuzzy Markov Model- UFMM

The UFMM is a set of hierarchical fuzzy Markov chains (top level fuzzy Markov chains - TFMC, lower level fuzzy Markov chains - LFMC). The TFMC represents the high-level operational units (states), associated connections (transitions) and transition fuzzy probabilities. And the various sub operations associated with an individual state of the TFMC could be modeled by LFMC. We consider a UFMM with the following assumptions for our proposed system.

- Each state of the TFMC is associated with a LFMC.
- Each LFMC consists a state from which the control is initiated (initial state) and a terminal state E.
- In the TFMC, each state is acted as an initial state and has a terminal state E.

E. Fuzzy Probist Fault Tree – FFT [2, 8]

It is a probist fault tree in which the probabilities of its basic events are represented by fuzzy sets or fuzzy numbers (fuzzy probability). And its logic functions are defined as

$$\text{AND operator function : } \tilde{p}_{\text{Top}} = \prod_{i=1}^n \tilde{p}_i^\alpha .$$

$$\text{OR operator function : } 1 - \prod_{i=1}^n (1 - \tilde{p}_i^\alpha)$$

Fuzzy Reliability of the top event:  $\tilde{R}_{\text{Top}} = 1 \ominus \tilde{p}_{\text{Top}}$ . Since the triangular fuzzy numbers are defined on [0, 1], instead of  $\alpha$ -cut arithmetic, we can use the fuzzy arithmetic operations [5].

III. FUZZY RELIABILITY OF A FUZZY MARKOV CHAIN

Consider a fuzzy Markov chain with the state space  $\{1, 2, \dots, n, S\}$ , where  $S$  is the terminal (success) state, 1 is the initial state. And it has the transition fuzzy probability matrix  $\tilde{P}$  of order  $(n+1)$ , i.e., there exists  $p_{ij} \in \tilde{p}_{ij}[1]$  such that  $\sum_{j=1}^{n,S} p_{ij} = 1$ . Since  $\tilde{p}_{Si} = 0$  and  $\tilde{p}_{SS} = 1$ ,  $S$  is an absorbing state.

A. Construction of imperfect Markov chain and Reliability Expression

To express the fuzzy reliability of the system in terms of the fuzzy reliability of its each state, we use the restricted fuzzy arithmetic on the transition fuzzy probability matrix  $\tilde{P}$ . Let  $\alpha \in [0,1]$  and  $\text{Dom}[\alpha]$  be the set of all  $p_{ij} \in \tilde{p}_{ij}[\alpha]$ ,  $i, j \in \{1, 2, \dots, n, S\}$  so that if we form a transition matrix  $P = (p_{ij})$  with these  $p_{ij}$  values, each row sum equals one. Let  $v = (p_{11}, p_{12}, \dots, p_{SS})$ . The row vector  $v$  is just all the  $p_{ij}$  in a transition matrix.  $P = (p_{ij})$ . Then  $\text{Dom}[\alpha]$  is all the vectors  $v$ . Hence each  $v \in \text{Dom}[\alpha]$  represents the transition probability matrix  $P$  associated to a finite Markov chain with state space  $\{1, 2, \dots, n, S\}$ . To find the reliability of this finite Markov chain, we construct an imperfect Markov chain as defined in section I.

Using the standard method of computing absorbing probabilities [3], the reliability of the Markov chain (the probability of ever reaching the absorbing state  $S$  from a transient state) can be expressed as,

$$R_L = \sum_{i=1}^n (I_n - \hat{Q})_{ii}^{-1} r_i p_{iS} \tag{1}$$

where  $\hat{Q}$  is the  $(n \times n)$  sub matrix of  $Q$  obtained by removing its last two rows and columns corresponding to  $S, F$  and  $I_n$  is the identity matrix. And  $R_L$  is the function of  $r_i$  and  $p_{ij}$

in which the  $p_{ij}$  values are known but the  $r_i$  values are unknown. Hence to find the values of  $r_i$ , we use the following linear programming technique as in [6].

B. Linear Programming Technique

Now we use a procedure for finding the system reliability in which we associate a positive integer  $k_i$  with its each individual state. Let  $X_i$  be the number of failures among  $k_i$ . And it is assumed that  $X_1, X_2, \dots, X_n$  are mutually independent random variables. We say that the system is reliable as long as  $X_i = 0$  for  $i = 1, 2, \dots, n$ .

Our objective is to find the optimum value for  $k_i$  such that the probability that the system reliability  $R_L$  is less than a threshold value  $R_0$  is below some specified small fraction  $\beta$ . When  $k_i$  is large,  $(1 - r_i)$  is small, we can approximate the distribution of  $X_i$  as a Poisson distribution with parameter  $k_i (1 - r_i)$ ,  $i = 1, 2, \dots, n$ . Thus, the probability for the system reliability is given by  $p(\sum_{i=1}^n X_i = 0) = \exp(-(\sum_{i=1}^n k_i (1 - r_i)))$ . And we can formulate the problem as,

$$\left. \begin{aligned} &\text{optimize } k_1 + k_2 + \dots + k_n \\ &\text{subject to} \\ &\exp(-(\sum_{i=1}^n k_i (1 - r_i))) \leq \beta \\ &\text{such that } 0 \leq r_i \leq 1, R_L \leq R_0 \end{aligned} \right\} \tag{2}$$

$k_i$  is a positive integer. We make the observation that (2) is equivalent to solving the following sub problem.

$$\left. \begin{aligned} &\text{Minimize } \sum_{i=1}^n k_i (1 - r_i) \\ &\text{subject to} \\ &0 \leq r_i \leq 1, R_L \leq R_0 \end{aligned} \right\} \tag{3}$$

and its optimum value will be greater than or equal to  $-\ln(\beta)$ . Thus, the overall problem can be rewritten as the following two-stage linear programming problem.

$$\left. \begin{aligned} &\text{optimize } \sum_{i=1}^n k_i \\ &\text{subject to} \\ &\text{Minimize } \sum_{i=1}^n k_i (1 - r_i) \\ &\text{such that } 0 \leq r_i \leq 1, R_L \leq R_0 \\ &\hspace{15em} \geq -\ln(\beta) \end{aligned} \right\} \tag{4}$$

We solve this two stage optimization problem using the following algorithm 1 [6].

C. Algorithm 1

In order to simplify the procedure, we ignore the integer restriction on  $k_i$  and we round up optimum continuous solution to get the final values for  $k_i$ .

1. For each  $i = 1, 2, \dots, n$  define a vector  $\alpha_i = (\alpha_{i1}, \alpha_{i2}, \dots, \alpha_{ij}, \dots, \alpha_{in})$  by setting the  $(i = j)^{th}$  element equal to  $r_i$  and all the other elements equal to 1. Solve for the value of this  $i^{th}$  element using  $R_L = R_0$ .
2. Use each of these  $n$  vectors to define  $n$  initial linear constraints of the form  $\sum_{j=1}^n k_j (1 - \alpha_{ij}) \geq -\ln(\beta)$ .

Define the initial linear programming problem as optimize  $\sum_{i=1}^n k_i$  subject to these  $n$  constraints and

$$k_i \geq 0. \text{ Let the solution be } K = (k_1, k_2, \dots, k_n).$$

3. Use the vector  $K$  to define the objective function of the non-linear optimization problem (3) and solve it. Then the optimum solution is given by  $r = (r_1, r_2, \dots, r_n)$ . If  $\sum_{j=1}^n k_j (1 - \alpha_{ij}) \geq -\ln(\beta)$ , then stop and  $K$  is optimal. Round up the values of  $k_i$ . Otherwise, proceed to the next step.

4. Redefine the current linear programming problem as

$$\left. \begin{array}{l} \text{optimize } \sum_{i=1}^n k_i \\ \text{subject to} \\ \sum_{i=1}^n k_i (1 - r_i) \geq -\ln(\beta) \\ 0 \leq r_i \leq 1, k_i \geq 0 \end{array} \right\} \quad (5).$$

This omits the current solution vector  $K$ , which is infeasible.

5. Solve the linear programming problem (5) and obtain a new solution vector and replace  $K$  with this new vector. Go to step 3

By this algorithm, we get optimum solution vector  $K = (k_1, k_2, \dots, k_n)$  and  $r = (r_1, r_2, \dots, r_n)$ . Using the solution vector  $r$ , we get  $R_L$ .

D. Algorithm 2

This algorithm is used to find the fuzzy reliability of the fuzzy Markov chain.

1. Define  $\text{Dom}[\alpha]$  for  $\alpha \in [0,1]$
2. For each  $v \in \text{Dom}[\alpha]$ , construct an imperfect Markov chain as described in subsection (A) of section II and model two stage optimization problem for this Markov chain as in subsection (B) of section III.
3. Solve the two stage optimization problem using algorithm 1 and compute  $R_L$  values.

4. Find the  $\alpha$  - cuts for the fuzzy reliability  $\tilde{R}_L$  as follows  $\tilde{R}_L[\alpha] = [R_{L_1}(\alpha), R_{L_2}(\alpha)]$  where  $R_{L_1}(\alpha) = \min\{R_L / R_L \text{ is the reliability of the Markov chain associated to } v \in \text{Dom}[\alpha]\}$   
 $R_{L_2}(\alpha) = \max\{R_L / R_L \text{ is the reliability of the Markov chain associated to } v \in \text{Dom}[\alpha]\}$
5. Using these  $\alpha$  - cuts of  $\tilde{R}_L$ , we can find the triangular fuzzy number to represent the fuzzy reliability  $\tilde{R}_L$  of the fuzzy Markov chain.

IV. FUZZY RELIABILITY OF A UFMM

Let  $\{1, 2, \dots, m\}$  be the state space for the TFMC in which each state is acting as an initial state. By algorithm 2, we can find the fuzzy reliability of each LFMC which is associated to the state of the TFMC. To find the fuzzy reliability of the TFMC, first we have to construct an imperfect fuzzy Markov chain  $\{1, 2, \dots, m, S, F\}$  where  $1, 2, \dots, m$  are the transient states and  $S, F$  are the absorbing states. The entries of its transition fuzzy probability Matrix  $\tilde{T}$  are obtained using the fuzzy arithmetic operations as follows.

$$\left. \begin{array}{l} \tilde{t}_{ij} = \tilde{r}_i \otimes \tilde{p}_{ij} \text{ for } i=1,2,\dots,m \text{ and } j=1,2,\dots,m,S \\ \tilde{t}_{iF} = 1 \oplus r_i \text{ for } i=1,2,\dots,m \\ \tilde{t}_{SS} = \tilde{t}_{FF} = (1,1,1) \\ \tilde{t}_{Sj} = \tilde{t}_{Fj} = (0,0,0) \text{ for } j=1,2,\dots,m \end{array} \right\} \quad (6)$$

We apply the restriction on  $\tilde{T}$  as explained in section three. For each  $w = (t_{11}, t_{12}, \dots, t_{FF}) \in \text{Dom}[\alpha]$ , we get an transition probability matrix  $T$ . And we define the probability that the system ever reaching the failure state from each of its states by the standard method of computing absorbing probabilities [3] as

$$P_F = (I_n - \hat{T})^{-1} \text{ [column corresponding to the failure state 'F']}$$

$$P_{iF} = \sum_{k=1}^m (I_n - \hat{T})_{ik}^{-1} t_{kF}$$

We get  $P_F = (p_{1F}, p_{2F}, \dots, p_{mF})$ . Then,

$$\tilde{P}_{iF}[\alpha] = [P_{iF_1}(\alpha), P_{iF_2}(\alpha)] \text{ for } i = 1, 2, \dots, m \text{ and}$$

$$P_{iF_1}(\alpha) = \min\{P_{iF} / P_{iF} \in P_F \text{ and } P_F \text{ corresponds to } w \in \text{Dom}[\alpha]\}$$

$$P_{iF_2}(\alpha) = \max\{P_{iF} / P_{iF} \in P_F \text{ and } P_F \text{ corresponds to } w \in \text{Dom}[\alpha]\}$$

Using these  $\alpha$  - cuts, we can find the fuzzy probability  $\tilde{p}_{iF}$  that the system ever enters the failure state from a state 'i' ( $i = 1, 2, \dots, m$ ) of the TFMC and  $\tilde{P}_F = (\tilde{P}_{1F}, \tilde{P}_{2F}, \dots, \tilde{P}_{mF})$ . For this TFMC, we construct a FFT by considering the failure of each of its states as the basic events of FFT. And

$\tilde{P}_F$  gives the fuzzy probability for these basic events. According to the system activities, we use logic operator function and get the fuzzy probability for the top event (failure of the system) as shown in subsection (E) of section I. Finally, the fuzzy reliability of an UFMM is given by  $\tilde{R}_U = 1 \ominus \tilde{P}_{Top}$  which is the fuzzy reliability of the proposed communication network.

V. FUZZY RELIABILITY OF A WEB PAGE

In this section, we consider the web page ‘Departments’ with three hyper links Information Technology (IT), Electronics & Communications Engineering (ECE) and Electrical and Electronics Engineering (EEE) of a web site www.ssn.edu.in. We model the web page as an UFMM whose TFMC has state space {IT, EEE, ECE} such that each state is an initial state, because each state is acted as a hyper link in the web page. And each state is associated with a fuzzy Markov chain (LFMC) whose states are H - home page, N - news and events, FA - faculty, I- infrastructure.

From the log files collected for a period, we extract the data, N - the total number of transitions from state ‘i’ to state ‘j’ and s - the number of successes among those transitions. Find the transition probability  $p_{ij} = (s/N)$ . We fuzzify this  $p_{ij}$ , using the formula,  $\left[ p - (1-\alpha) \sqrt{\frac{p(1-p)}{N}}, p + (1-\alpha) \sqrt{\frac{p(1-p)}{N}} \right]$  for  $\alpha \in [0,1]$ . And they are shown in Fig. 1, Fig. 2, Fig. 3, Fig. 4.

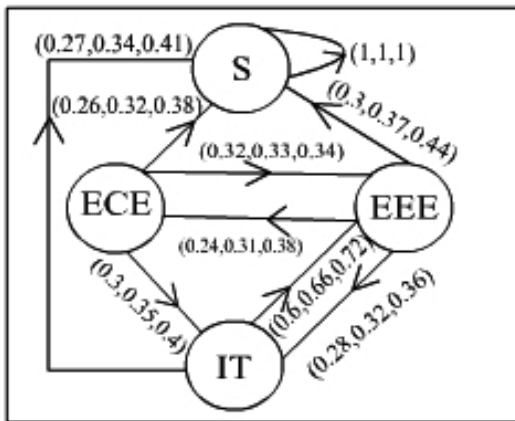


Figure 1. Top Level Fuzzy Probabilistic Markov Model

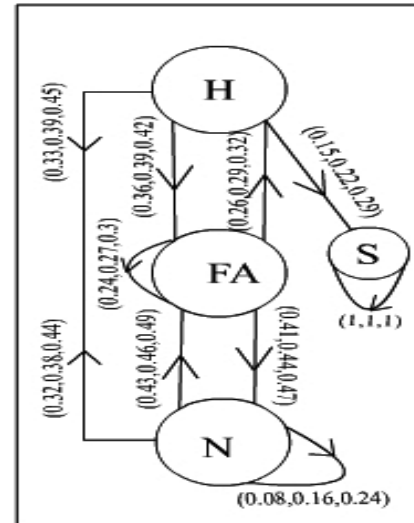


Figure 2. Lower Level Fuzzy Probabilistic Markov Model for ECE

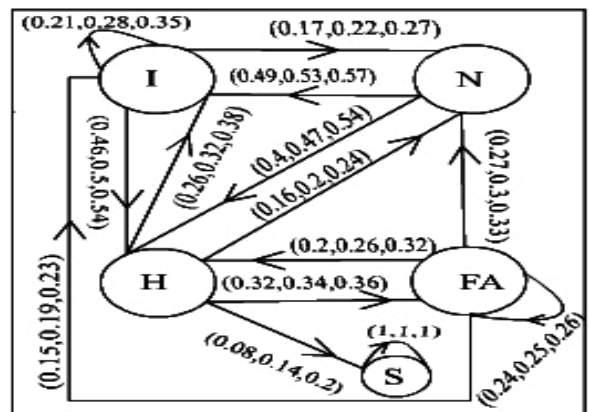


Figure 3. Lower Level Fuzzy Probabilistic Markov Model for EEE

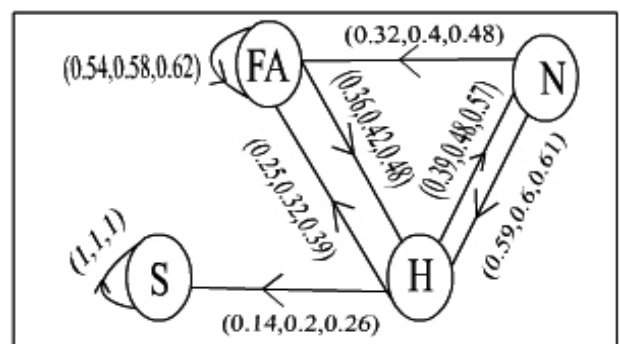


Figure 4. Lower Level Fuzzy Probabilistic Markov Model for IT

These transition fuzzy probabilities  $\tilde{p}_{ij}$  form the transition fuzzy probability matrix  $\tilde{P}$ . Here we get three transition fuzzy probability matrices  $\tilde{P}_{IT}$ ,  $\tilde{P}_{EEE}$ ,  $\tilde{P}_{ECE}$  for the LFMCs associated to the states IT, EEE and ECE and  $\tilde{P}_D$  for the TFMC ‘Departments’. We apply the restriction on  $\tilde{P}_{IT}$ ,  $\tilde{P}_{EEE}$ ,  $\tilde{P}_{ECE}$ . We use ‘C’ coding to get the reliability expression (1) and MATLAB to solve linear, non linear

programming problems. In (4), we assume  $R_0 = 0.96$ ,  $\beta = 0.05$ . Using algorithm 2, we get the fuzzy reliability  $\tilde{R}_{IT}, \tilde{R}_{EEE}, \tilde{R}_{ECE}$  for the states of TFMC. The calculated fuzzy reliability values are given in the following table.

TABLE I.  
FUZZY RELIABILITY Of LFMCS

	$\alpha = 0$	$\alpha = 1$	Max Hits
$\tilde{R}_{IT}$	[ 0.92, 0.96]	0.94	750
$\tilde{R}_{ECE}$	[ 0.91, 0.96 ]	0.95	600
$\tilde{R}_{EEE}$	[ 0.88, 0.95 ]	0.93	1200

To find the fuzzy probist reliability of the UFMM, we have to construct a fuzzy probist fault tree whose basic events are A, B, C and its top event is D where A-the event that the system (web page) enters the state F from IT, B-the event that the system (web page) enters the state F from EEE, C-the event that the system (web page) enters the state F from ECE, D-the event of the system (web page) failure. We construct an imperfect fuzzy Markov chain for the TFMC. The entries of its transition fuzzy probability matrix are obtained using (6). Its transition fuzzy probability matrix  $\tilde{T}$  is

	IT	EEE	ECE	S	F
IT	0	(0.55,0.62,0.69)	0	(0.25,0.32,0.39)	(0.04,0.06,0.08)
EEE	(0.25,0.3,0.34)	0	(0.21,0.29,0.36)	(0.26,0.34,0.42)	(0.05,0.07,0.12)
ECE	(0.27,0.33,0.38)	(0.29,0.31,0.33)	0	(0.24,0.3,0.36)	(0.04,0.06,0.09)
S	0	0	0	1	0
F	0	0	0	0	1

As explained in the previous section, we compute  $\tilde{P}_F = (\tilde{P}_{IT,F}, \tilde{P}_{EEE,F}, \tilde{P}_{ECE,F})$  as given below.

TABLE II.  
FUZZY PROBABILITY Of BASIC EVENTS

	$\alpha = 0$	$\alpha = 1$
$\tilde{P}_{IT, F}$	[ 0.11, 0.2 ]	0.16
$\tilde{P}_{ECE, F}$	[0.11, 0.24 ]	0.16
$\tilde{P}_{EEE, F}$	[0.13, 0.22 ]	0.16

We can say that a web page is failure if it is not able to access any of its links. i.e., the system ever enters the failure state from IT, EEE and ECE. Hence we construct a FFT with AND gate. It is depicted in Fig. 5.

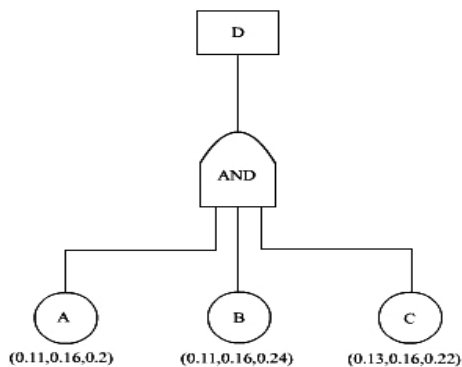


Figure 5. Fuzzy Probist Fault Tree

Then  $\tilde{P}_D = (0.001573, 0.004096, 0.01056)$  and  $\tilde{R}_D = (0.98944, 0.995904, 0.998427)$ . This implies that the web page 'Departments' is more reliable with the maximum number of two thousand five hundred and fifty hits.

CONCLUSIONS

We have considered a communication network with different components such that each component has sub operations. And its behavior (transition between components) follow Markovian property and it is being measured using fuzzy probability. To find its fuzzy probist reliability, we have viewed the network as UFMM. We have used linear programming technique with restricted arithmetic and FFT. We have illustrated this method through finding the fuzzy probist reliability of the web page 'Departments' of the website www.ssn.edu.in as (0.99,1,1). The complexity in the computation of the fuzzy reliability of the LFMCS is ruled out by the 'C' codes for finding the reliability expression and the MATLAB tool for solving the linear and non linear programming problems occurs in algorithm 1.

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